

**INTEGER ALGORITHMS TO SOLVE
LINEAR EQUATIONS AND SYSTEMS**

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Abstract. Original integer general solutions, together with examples, are presented to solve linear equations and systems.

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Introduction.

The present work includes some of the author's original researches on the integer solutions of equations and linear systems:

1. The notion of "general integer solution" of a linear equation with two unknowns is extended to linear

equations with n unknowns and then, to linear systems.

2. The properties of the general integer solution are determined (both of a linear and system).

3. Seven original integer algorithms (two for linear equations and five for linear systems) are presented. The algorithms are strictly demonstrated and an example for each of them is given. These algorithms can be easily introduced in a computer.

INTEGER SOLUTIONS OF LINEAR EQUATIONS

Definitions and properties of the integer solution of linear equations (1.e):

Let 1.e:

$$(1) \quad \sum_{i=1}^n a_i x_i = b, \text{ with not all } a_i \text{ null and } b \text{ from } \mathbb{Z}.$$

Again, let $h \in \mathbb{N}$, and the functions $f_i : \mathbb{Z}^h \rightarrow \mathbb{Z}$,
 $i = \overline{1, n}$.

Definition 1

$x_i = \overline{x_i}, \quad i = \overline{1, n}$, is the particular integer solution of equation (1), if all $x_i \in \mathbb{Z}$ and $\sum_{i=1}^n a_i x_i = b$.

Definition 2

$x_i = f_i(k_1, \dots, k_h)$, $i = \overline{1, n}$, is the general integer solution of equation (1) if:

$$(a) \sum_{i=1}^n a_i f_i(k_1, \dots, k_h) = b \quad \forall (k_1, \dots, k_h) \in \mathbb{Z}^h,$$

(b) Irrespectively of $f_i(x_1, \dots, x_n)$ there is a particular integer solution for (1) $(k_1, \dots, k_h) \in \mathbb{Z}^h$ so

that $x_i = f_i(k_1, \dots, k_h)$ for all $i = \overline{1, n}$.

We will further see that the general integer solution can be expressed by linear functions.

We consider for $1 \leq i \leq n$, the functions $f_i = \sum_{j=1}^h c_{ij} k_j + d_i$ with all $c_{ij}, d_i \in \mathbb{Z}$.

Definition 3

$A = (c_{ij})_{i,j}$ is referred to as the matrix associated to the general solution of equation (1).

Definition 4

The integers k_1, \dots, k_s , $1 \leq s \leq h$, are independent if all the corresponding column vectors of matrix A are linearly independent.

Definition 5

An integer solution is s - times undetermined if the maximal number of independent parameters is s .

Theorem 1. The general integer solution of equation (1) is $(n-1)$ - times undetermined.

Proof

We suppose that the particular integer solution is of the form:

$$(2) \quad x_i = \sum_{e=1}^r i_{ie} p_e + v_i, \quad i = \overline{1, n}, \quad \text{with all } u_{ie}, v_i \in \mathbb{Z},$$

p_e = are parameters from \mathbb{Z} , while $a \leq r < n - 1$.

Let (x_1, \dots, x_n) a general integer solution of equation

(1) (we are not interested in the case when the equation does not accept integer solution). The solution

$$x_j = a_n l_j + x_j^0 \quad \text{for } j = \overline{1, n-1}$$

$$x_n = - \left(\sum_{j=1}^{n-1} a_j k_j + x_n^0 \right)$$

is $(n - 1)$ - times undetermined (it can be easily checked that the order of the associated matrix is $n - 1$). Hence,

there are $n - 1$ undetermined solutions. Let, in the general case, a solution $n - 1$ times be undetermined:

$$x_i = \sum_{j=1}^{n-1} c_{ij}k_j + d_i \quad i = 1, n, \text{ with all } c_{ij}, d_i \in \mathbb{Z}.$$

The case when $b = 0$.

$$\text{Then } \sum_{i=1}^n a_i x_i = 0. \quad \text{It follows } \sum_{i=1}^n a_i x_i =$$

$$\begin{aligned} &= \sum_{i=1}^n a_i \left(\sum_{j=1}^{n-1} c_{ij}k_j + d_i \right) = \sum_{i=1}^n a_i \sum_{j=1}^{n-1} c_{ij}k_j + \sum_{i=1}^n a_i d_i \\ &= 0. \end{aligned}$$

$$\text{For } k_{j_0} = 1 \text{ and } k_j = 0, j \dots j_0, \text{ it follows } \sum_{i=1}^n a_i c_{ij_0} = 0.$$

Let the homogenous linear system of n equations with n unknowns be:

$$\sum_{i=1}^n x_i c_{ij} = 0, \quad j = 1, n - 1;$$

$$\sum_{i=1}^n x_i d_i = 0$$

which, obviously accepts the solution $x_i = a_i, i = 1, n$, different from the trivial one. Hence the determinant of

the system is zero, i.e., the vectors $C_j = (c_{1j}, \dots, c_{nj})^t$,

$j = 1, n - 1$, $D = (d_1, \dots, d_n)^t$ are linearly dependent.

But the solution being $n - 1$ times undetermined it follows

that C_j , $j = 1, n - 1$, are linearly independent. Then (C_1, \dots, C_{n-1}) determines a free sub-module Z of the order $n - 1$ in Z_n of solutions for the given equation.

Let us see what can be obtained from (2). We have:

$$0 = \sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i \left(\sum_{e=1}^r u_{ie} P_e + v_i \right). \text{ As above,}$$

we obtain:

$$\sum_{i=1}^n a_i v_i = 0 \quad \text{and} \quad \sum_{e=1}^r a_i u_{ie} = 0; \text{ and similarly, the}$$

vectors $U_h = (u_{1h}, \dots, u_{nh})$ are linearly independent,

$h = 1, r$. U_h , $h = 1, r$, and $V = (v_1, \dots, v_n)$ are particular integer solutions of the homogenous linear equation.

Subcase (a1)

U_h , $h = 1, r$, are linearly dependent. It gives $\{U_1, \dots, U_r\}$ = the free submodule of order r in Z^n of solutions of the equation. Hence, there are solutions

from $\{V_1, \dots, V_{n-1}\}$ which are not from $\{U_1, \dots, U_r\}$, this contradicts the fact that (2) is the general integer solution.

Subcase (a2)

$U_h, h = 1, r, V$ are linearly independent. Then, $\{U_1, \dots, U_r\} + V$ is a linear variety of dimension $< n - 1 = \dim \{V_1, \dots, V_{n-1}\}$ and the conclusion can be similarly drawn. The case when $b \neq 0$.

So, $\sum_{i=1}^n a_i x_i = b$. Then $\sum_{i=1}^n a_i (\sum_{j=1}^{n-1} c_{ij} k_j + d_i) =$

$$= \sum_{j=1}^{n-1} (\sum_{i=1}^n a_i c_{ij}) k_j + \sum_{i=1}^n a_i d_i = b, V(k_1, \dots, k_{n-1}, 0) \in \mathbb{Z}^{n-1}.$$

As in the previous case we get $\sum_{i=1}^n a_i d_i = b$ and

$\sum_{i=1}^n a_i c_{ij} = 0, V_j = 1, n - 1$. The vectors $C_j = (c_{1j}, \dots, c_{nj})^t, j = 1, n - 1$, are linearly independent because the solution is $n - 1$ times undetermined.

Conversely, if C_1, \dots, C_{n-1}, D (where $D = (d_1, \dots, d_n)^t$)

were linearly dependent, it would mean that $D = \sum_{j=1}^{n-1} s_j C_j$,

with all s_j scalar; it would also mean that $b = \sum_{i=1}^n a_i d_i =$

$$= \sum_{i=1}^n a_i \left(\sum_{j=1}^{n-1} s_j c_{ij} \right) = \sum_{j=1}^{n-1} s_j \left(\sum_{i=1}^n a_i c_{ij} \right) = 0. \text{ Impossibly,}$$

(3) then $\{C_1, \dots, C_{n-1}\} + D$ is a linear variety.

Let us see what we can obtain from (2). We have:

$$b = \sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i \left(\sum_{e=1}^r u_{ie} P_e + v_i \right) = \sum_{e=1}^r \left(\sum_{i=1}^n a_i u_{ie} \right) P_e$$

$$+ \sum_{i=1}^n a_i v_i$$

and, similarly: $\sum_{i=1}^n a_i v_i = b$ and $\sum_{i=1}^n a_i u_{ie} = 0, \forall e = 1, r,$ ---

respectively. The vectors $U_e = (u_{1e}, \dots, u_{ne})^t, e = 1, r,$ ---

are linearly independent because the solution is r - times undetermined.

A procedure like that applied in (3) gives that U_1, \dots, U_r, V are linearly independent, where $V = (v_1, \dots, v_n)^t.$ Then $\{U_1, \dots, U_r\} + V =$ a linear variety = free sub-module of order $r < n - 1.$ That is, we can find vectors from $C_1, \dots, C_{n-1} + D$ which are not from $\{U_1, \dots, U_r\} + V,$ contradicting the "general" characteristic

of the integer number solution. Hence, the general integer solution is $n - 1$ times undetermined.

Theorem 2. The general integer solution of the

homogenous linear equation $\sum_{i=1}^n a_i x_i = 0$ (all $a_i \in \mathbb{Z} \setminus \{0\}$)

can be written under the form:

$$(4) \quad x_i = \sum_{j=1}^{n-1} c_{ij} k_j, \quad i = 1, n, \quad (\text{with } d_1 = \dots = d_n = 0).$$

Definition 6. This is called the standard form of the general integer solution of a homogenous linear equation.

Proof

We consider the general integer solution under the

form: $x_i = \sum_{j=1}^{n-1} c_{ij} p_j + d_i, \quad i = 1, n,$ with not all $d_i = 0.$ We show that it can be written under the form (4). The homogenous equation admits the trivial solution

$x_i = 0,$ for

$i = 1, n.$ There is $(p_1, \dots, p_{n-1}) \in \mathbb{Z}^{n-1}$ so that $\sum_{j=1}^{n-1} c_{ij} p_j +$

$+ d_i = 0, \quad \forall i = 1, n.$ Substituting: $p_j = k_j + p_j,$

$j = 1, n - 1,$ in the form from the beginning of the proof we obtain form (4). We have to

mention that the substitution does not diminish the

degree of -----

generality as $P_j \in \mathbb{Z}$ because $k_j \in \mathbb{Z}$, $j \in 1, n - 1$.

Theorem 3. The general integer solution of a non-homogeneous linear equation of its associated homogenous linear equation + any particular integer solution of the non-homogeneous linear equation.

Proof

Let $x_i = \sum_{j=1}^{n-1} c_{ij}k_j$, $i = 1, n$, be the general integer solution of the associated homogenous linear equation and,

again, let $x_i = v_i$, $i = 1, n$, be a particular integer solution of the non-homogeneous linear equation. Then,

$$x_i = \sum_{j=1}^{n-1} c_{ij}k_j +$$

+ v_i , $i = 1, n$, is the general integer solution of the

non-homogeneous linear equation. Actually, $\sum_{i=1}^n a_i x_i =$

$$= \sum_{i=1}^n a_i \left(\sum_{j=1}^{n-1} c_{ij}k_j + v_i \right) = \sum_{i=1}^n a_i \left(\sum_{j=1}^{n-1} c_{ij}k_j \right) + \sum_{i=1}^n a_i v_i \\ = b;$$

if $x_i = x_i^0$, $i = 1, n$, is a particular integer solution of

the non-homogeneous linear equation, then $x_i = x_i - v_i$,

 $i = 1, n$, is a particular integer solution of the
 homogenous linear

equation; hence, there is $(k_1, \dots, k_{n-1}) \in \mathbb{Z}^{n-1}$ so that

$\sum_{j=1}^{n-1} c_{ij} k_j = x_i - v_i, \forall i = 1, n$, i.e., $\sum_{j=1}^{n-1} c_{ij} k_j + v_i = x_i, \forall i = 1, n$

 $1, n$, what had to be proven.

Theorem 4. If $x_i = \sum_{j=1}^{n-1} c_{ij} k_j$, $i = 1, n$, is the

general integer solution of a homogenous linear equation
 $(c_{ij}, \dots, c_{nj}) = 1$,

 $\forall j = 1, n - 1$.

The demonstration is made by reductio ad absurdum.
 If $\exists j_0$, $1 \leq j_0 \leq n - 1$,

so that $(c_{1j_0}, \dots, c_{nj_0}) = d_{j_0} \neq \pm 1$,

then $c_{ij_0} = c_{ij_0} \overline{d_{ij_0}}$ with $(c_{ij_0}, \dots, c_{nj_0}) = 1$, $\forall i = 1, n$.

 But $x_i = c_{ij_0}$, $i = 1, n$, represents a particular integer

solution as $\sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i c_{ij_0} = 1/d_{j_0}$! $\sum_{i=1}^n a_i c_{ij_0} = 0$

 (because $x_i = c_{ij_0}$, $i = 1, n$, is a particular integer
 solution from the general integer solution by
 introducing $k_j = 1$ and

$k_j = 0, j \neq j_0$). But the particular integer solution

$x_i = c_{ij_0}, i = 1, n$, cannot be obtained, by introducing

whole number parameters (as it should), from the general integer solution, as, from the linear system of n equations and $n - 1$ unknowns, which is compatible, we obtain:

$$x_i = \sum_{\substack{j=1 \\ j \neq j_0}}^n c_{ij} k_j + c_{ij_0} d_{j_0} k_{j_0} = c_{ij_0}, i = 1, n,$$

Leaving aside the last equation--which is a linear combination of the other $n - 1$ equations, a Kramerian system is obtained. It follows up:

$$k_{j_0} = \frac{\left| \begin{array}{c} c_{11} \dots c_{1j_0} \dots c_{1,n-1} \\ \vdots \\ c_{n-1,1} \dots c_{n-1,j_0} \dots c_{n-1,n-1} \end{array} \right|}{\left| \begin{array}{c} c_{11} \dots c_{1j_0} d_{j_0} \dots c_{1,n-1} \\ \vdots \\ c_{n-1,1} \dots c_{n-1,j_0} d_{j_0} \dots c_{n-1,n-1} \end{array} \right|} = \frac{1}{d_{j_0}} \in \mathbb{Z}.$$

The assumption is false and thus the proof results.

Theorem 5. Considering the equation (1) with $(a_1, \dots, a_n) = 1$, $b = 0$, and the general integer solution

$$x_i = \sum_{j=1}^{n-1} c_{ij} k_j, i = 1, n, \text{ then } (a_1, \dots, a_{i-1}, a_{i+1}, \dots,$$

$a_n) = (c_{i1}, \dots, c_{in-1})$, $\forall i = 1, n$. The demonstration is made by

double divisibility. Let i_0 , $1 \leq i_0 \leq n$, be arbitrary but

fixed. $x_i = \sum_{j=1}^{n-1} c_{ij} k_j$. Consider the equation $\sum_{i \neq i_0} a_i x_i$

$= -a_{i_0} x_{i_0}$. We have shown that $x_i = c_{ij}$, $i = 1, n$ is a particular integer solution irrespective of j , $a \leq j \leq n - 1$. The equation $\sum_{i \neq i_0} a_i x_i = -a_{i_0} c_{i_0 j}$, obviously,

admits the integer solution $x_i = c_{ij}$, $i \neq i_0$. Then

$(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ divides $-a_{i_0} c_{i_0 j}$, but, as we

have assumed that $(a_1, \dots, a_n) = 1$, it follows that

$(a_1, \dots, a_{i_0-1}, a_{i_0+1}, \dots, a_n) \mid c_{i_0 j}$, irrespective of j .

Hence $(a_1, \dots, a_{i_0-1}, a_{i_0+1}, \dots, a_n) \mid (c_{i_0 1}, \dots, c_{i_0 n-1})$,

$\forall i_0 \in 1, n$, and the divisibility in one sense was proven.

Inverse Divisibility

Let us suppose the contrary and say that $\forall i_1 \in 0$

$\forall i_1 \in 1, n$ for which $(a_1, \dots, a_{i_1-1}, a_{i_1+1}, \dots, a_n) = d_{i_1} \dots d_{i_2} \dots (c_{i_1 1}, \dots, c_{i_1 n-1})$; we have considered $d_{i_1 1}$ and $d_{i_1 2}$ without restricting the generality. $d_{i_1} \mid d_{i_2}$ according

to

1 1

the first part of the demonstration. Hence, $\rightarrow d \neq 0$ so that

$$d_{i_1 2} = d \neq d_{i_1 1}, |d| \neq 1.$$

$$x_i = \sum_{j=1}^{n-1} c_{i j} k_j = d \neq d_{i_1 1} \sum_{j=1}^{n-1} c_{i j} k_j; \sum_{i=1}^n a_i x_i = 0 \text{ and}$$

$$a_i x_i =$$

$$= -a_{i_1} x_{i_1} \sum_{i=i_1}^n a_i x_i = -a_{i_1} d \neq d_{i_1 1} \sum_{j=1}^{n-1} c_{i_1 j} k_j, \text{ where}$$

$$(c_{i_1 1}, \dots, c_{i_1 n-1}) \neq 1$$

The nonhomogeneous linear equation $\sum_{i=i_1}^n a_i x_i = -a_{i_1} d_{i_1 1}$ admits

integer solution because $a_{i_1} d_{i_1 1}$ is divisible by

$(a_1, \dots, a_{i_1-1}, a_{i_1+1}, \dots, a_n)$. Let $x_i = x_i^0$, $i \neq i_1$ be its

particular integer solution. It follows that the equation

$$\sum_{i=1}^n a_i x_i = 0 \text{ admits the particular solution } x_i = x_i^0,$$

$x_i = d_i$, which is written as (5). We show that

(5)

cannot be obtained from the general solution by integer number parameters:

$$\sum_{j=1}^{n-1} c_{ij} k_j = x_i, \quad i \dots i_1;$$

$$d_1 \dots d_{i_1} \sum_{j=1}^{n-1} c_{ij} k_j = d_{i_1} \quad (6)$$

But equation (6) does not admit integer solution because $d_1 \dots d_{i_1}$ thus, contradicting, thus, the "general" . characteristic of the integer solution.

As a conclusion we can write:

Theorem 6. Let the homogenous linear equation be:

$$\sum_{i=1}^n a_i x_i = 0 \text{ with all } a_i \in \mathbb{Z} \setminus \{0\}, \text{ and } (a_1, \dots, a_n) \neq 1.$$

Let $x_i = \sum_{j=1}^h c_{ij} k_j$, $i = 1, n$ be, with all $c_{ij} \in \mathbb{Z}$, all k_j

whole integer parameters and $h \in \mathbb{N}$, a general integer solution of the equation. Then,

1° the solution is $n - 1$ times undetermined;

2° $V_j = 1, n - 1$ we have $(c_{1j}, \dots, c_{nj}) = 1$;

3° $V_i = 1, n$ we have $(c_{i1}, \dots, c_{in-1}) = (a_1, \dots, a_{i-1},$

a_{i+1}, \dots, a_n). The proof results from Theorems 1, 4 and 5.

Note 1. The only equation of the form (1) which is $n -$ times undetermined is the trivial equation $0 \neq x_1 + \dots + 0 \neq x_n = 0$.

Note 2. The converse of theorem 6 is not true.

Counterexample:

$$\begin{aligned} x_1 &= -k_1 + k_2 \\ x_2 &= 5k_1 + 3k_2 \\ x_3 &= 7k_1 - k_2, \quad k_1, k_2 \in \mathbb{Z} \end{aligned} \tag{7}$$

is not the general integer solution of the equation

$$-13x_1 + 3x_2 - 4x_3 = 0 \tag{8}$$

although the solution (7) verifies the points $1^0, 2^0$ and 3^0

of theorem 6. $(1, 7, 2)$ is the particular integer solution

of (8) but cannot be obtained by introducing integer number

parameters in (7) because from

$$-k_1 + k_2 = 1$$

$$5k_1 + 3k_2 = 7$$

$$7k_1 - k_2 = 2$$

follows that $k = 1/2 \ 0 \ Z$ and $k = 3/2 \ 0 \ Z$ (unique roots).

Reference:

[1] Smarandache, Florentin, "Whole number solutions of linear equations and systems", M. Sc. Thesis, 1979, University of Craiova (under the supervision of Assoc. Prof. Dr. Alexandru Dincă).

AN INTEGER NUMBER ALGORITHM
TO SOLVE LINEAR EQUATIONS

An algorithm is given which ascertains whether a linear equation admits integer number solutions or not; if it does, the general integer solution is determined.

Input

A linear equation $a_1x_1 + \dots + a_nx_n = b$ with $a_i, b \in \mathbb{Z}$,
 x_i being integer number unknowns, $i = 1, n$ and not all $a_i = 0$.

Output

Decision on the integer solution of this equation; if the equation has solutions in \mathbb{Z} , its general solution is obtained.

Method

- Step 1. Calculate $d = (a_1, \dots, a_n)$.
- Step 2. If d/b then "the equation has integer solution"; go on to Step 3. If d/b then "the equation does not admit integer solution"; stop.
- Step 3. Consider $h = 1$. If $|d| \dots 1$ divides the

equation by d ; consider $a_i: = a_i/d$, $i = 1, n$, $b: = b/d$.
 Step 4. Calculate $a = \min |a_s|$ and determine an i so
 $a_s \dots 0$

that $a_i = a$.

Step 5. If $a \dots 1$, go on to Step 7.

Step 6. If $a = 1$, then:

(A) $x_i = -(a_1x_1 + \dots + a_{i-1}x_{i-1} + a_{i+1}x_{i+1} + \dots + a_nx_n - b) / a_i$

(B) Substitute the value of x_i in the values of the other determined unknowns.

(C) Substitute integer number parameters for all the variables of the unknown values in the right term: k_1, k_2, \dots, k_{n-2} and k_{n-1} , respectively.

(D) Write down the general solution thus determined; stop.

Step 7. Write down all a_j , $j \dots i$ and b under the form:

$$a_j = a_i q_j + r_j$$

$$b = a_i q + r, \text{ where } q_j = \frac{a_j}{-a_i}, \quad q = \frac{b}{-a_i}$$

Step 8. Write $x_i = -q_1x_1 - \dots - q_{i-1}x_{i-1} - q_{i+1}x_{i+1} - \dots - q_nx_n + q - t_h$. Substitute the value of x_i in the values of the other determined unknowns.

Step 9. $a_1 := 4_1$ and $a_i := -a_i$
 Consider $\begin{array}{l} \cdot \\ \cdot \\ \cdot \\ a_{i-1} := r \\ a_{i+1} := r_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ a_n := r_n \end{array}$ and go back to Step 4.

Lemma 1. The previous algorithm is finite.

Proof

Let the initial linear equation be $a_1x_1 + \dots + a_nx_n = b$ with not all $a_i = 0$; it is considered that $\min_{s \neq 0} |a_s| = a_s$

$a_1 \dots 1$ (if not, it is renumbered). Following the algorithm, once we pass from this initial equation to a new equation: $a_1t_1 + a_2x_2 + \dots + a_nx_n = b'$, with $|a_1| < |a_i|$ for $i = 2, n$, $|b'| < |b|$ and $a_1 = -a_1$. It follows that $\min_{s \neq 0} |a_s| < \min_{s \neq 1} |a_s|$. We continue similarly and after a finite number of steps we get, at Step 4, at $a := 1$ (as, every time, at this step the actual a is strictly smaller than the previous a , according to the former note) and in this case the algorithm terminates.

Lemma 2. Let the linear equation be: (25) $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ with $\min_{s \neq 0} |a_s| = a_1$ and the

equation: (26) $- a_1 t_1 + r_2 x_2 + \dots + r_n x_n = r$ with $t_1 =$
 $= - x_1 - q_2 x_2 - \dots - q_n x_n + q$ where $r_i = a_i - a_i q_i$, $i =$
 $= 2, n$, $r = b - a_1 q$ while $q_i = \frac{a_i}{-a} - \frac{b}{-a_1}$. Then
 $x_1 = x_1^0, x_2 = x_2^0, \dots, x_n = x_n^0$ is particular solution of
equation (25) if and only if $t_1 = t_1^0 = - x_1^0 - q_2 x_2^0 - \dots -$
 $q_n x_n^0 + q$, $x_2 = x_2^0, \dots, x_n = x_n^0$ is a particular solution of
equation (26).

Proof

$x_1 = x_1^0, x_2 = x_2^0, \dots, x_n = x_n^0$ is a particular solution
of equation (25)] $a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$] $a_1 x_1 +$
 $+ (r_2 + a_1 q_2) x_2 + \dots + (r_n + a_1 q_n) x_n = a_1 q + r$] $r_2 x_2 +$
 $+ \dots + r_n x_n - a_1 (-x_1 - q_2 x_2 - \dots - q_n x_n + q) = r$] $- a_1 t_1 +$
 $+ r_2 x_2 + \dots + r_n x_n = r$] $t_1 = t_1^0, x_2 = x_2^0, \dots, x_n = x_n^0$ is a
particular solution of equation (26).

Lemma 3. $x_i = c_{i1} k_1 + \dots + c_{in-1} k_{n-1} + d_i$, $i = 1, n$ is
the general solution of equation (25) if and only if:
(28)

$$\begin{aligned}
t_1 &= - (c_{11} + q_2 c_{21} + \dots + q_n c_{n1}) k_1 - \dots - (c_{1n-1} + \\
&+ q_2 c_{2n-1} + \dots + q_n c_{nn-1}) k_n - (d_1 + q_2 d_2 + \dots + q_n d_n) + \\
&+ q, \quad x_j = c_{cj1} k_1 + \dots + c_{jn-1} k_{n-1} + d_j, \quad j = 2, n
\end{aligned}$$

is a general solution for equation (26).

Proof

$t_1 = t_1 = -x_1 - q_2x_2 - \dots - q_nx_n + q, x_2 = x_2, \dots, \dots, x_n = x_n$ is a particular solution of the equation (25)
] $x_1 = x_1, x_2 = x_2, \dots, x_n = x_n$ is a particular solution of equation (26)] $\rightarrow k_1 = k_1 \text{ } 0 \text{ } Z, \dots, k_n = k_n \text{ } 0 \text{ } Z$ so that

$$\begin{aligned}
 & x_i = c_{i1}k_1 + \dots + c_{in-1}k_{n-1} + d_i = x_i, \quad i = 1, n \quad \Rightarrow \quad k_1 = \\
 & = k_1 \in \mathbb{Z}, \quad \dots, \quad k_{n-1} \in \mathbb{Z} \quad \text{so that} \quad x_1 = c_{11}k_1 + \dots + c_{1n-1}k_{n-1} + \\
 & + d_1 = x_1, \quad i = 2, n \quad \text{and} \quad t_1 = - (c_{11} + q_2c_{21} + \dots + q_nc_{n1})k_1 - \\
 & - \dots - (c_{1n-1} + q_2c_{2n-1} + \dots + q_nc_{nn-1})k_{n-1} - (d_1 + q_2d_2 + \\
 & + \dots + q_nd_n) + q = - x_1 - q_2x_2 - \dots - q_nx_n + q = t_1.
 \end{aligned}$$

Lemma 4. The linear equation (29) $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ with $|a_1| = 1$ has the general solution: (30)

$$x_1 = - (a_2 k_2 + \dots + a_n k_n - b) a_1,$$

$$x_i = k_i \circ z,$$

$$i = 2, n.$$

Proof

Let $x_1 = x_1, x_2 = x_2, \dots, x_n = x_n$ be a particular solution of the equation (29). $\rightarrow k_2 = x_2, \dots, k_n = k_n$ so

that $x_1 = - (a_2x_2 + \dots + a_nx_n - b) / a_1 = x_1$, $x_2 = x_2$, \dots ,
 \dots , $x_n = x_n$.

Lemma 5. Let the linear equation be $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$, with $\min_{a_s \neq 0} |a_s| = a_1$ and $a_i = a_1q_i$, $i = 2, n$.

Then, the general solution of the equation is:

$$\begin{aligned} x_1 &= - (q_2k_2 + \dots + q_nk_n - q), \\ x_i &= k_i \quad 0 \quad \mathbb{Z}, \\ &\quad \dots \\ i &= 2, n. \end{aligned}$$

Proof

Dividing the equation by a_1 the conditions of Lemma 4 are met.

Theorem of Correctness. The preceding algorithm correctly calculates the general solution of the linear equation $a_1x_1 + \dots + a_nx_n = b$ with not all $a_i = 0$.

Proof

The algorithm is finite according to Lemma 1. The correctness of steps 1, 2, and 3 is obvious. At step 4 there is always $\min_{a_s \neq 0} |a_s|$ as not all $a_i = 0$. The correctness of substep 6(A) results from Lemmas 4 and 5,

respectively. This algorithm represents a procedure of obtaining the general solution of the initial equation by means of the general solutions of the linear equation obtained after the algorithm was followed several times (according to Lemmas 2 and 3); from Lemma 3 it follows that to obtain the general solution of an initial linear equation is equivalent to calculate the general solution of an equation at step 6(A), equations whose general solution is given in algorithm (according to Lemmas 4 and 5). The theorem of correctness has been fully proven.

Note. At step 4 of the algorithm we consider

$a := \min_{a_s \neq 0} |a_s|$ so that the number of iterations be as small

as possible. The algorithm works if we consider

$a := |a_i| \dots \max_{s=1, n} |a_s|$ but it takes longer. The algorithm

can be introduced in the computer.

Application

Calculate the integer solution of the equation:

$$6x_1 - 12x_2 - 8x_3 + 22x_4 = 14$$

Solution

The former algorhythmus is applied.

$$1. \quad (6, -12, -8, 22) = 2$$

2. $2|14$ so that the solution of the equation is in

\mathbb{Z}

3. $h: = 1$

$|2| \dots 1$; dividing the equation by 2 we get:

$$3x_1 = 6x_2 - 4x_3 + 11x_4 = 7$$

4. $a: = \min \{|3|, |-6|, |-4|, |11|\} = 3, i = 1$

5. $a \dots 1$

$$-6 = 3 \cdot (-2) + 0$$

$$-4 = 3 \cdot (-2) + 2$$

$$11 = 3 \cdot 3 + 2$$

$$7 = 3 \cdot 2 + 1$$

8. $x_1 = 2x_2 + 2x_3 - 3x_4 + 2 - t_1$

9. $a_2: = 0 \quad a_1: = -3$

$$a_3: = 2 \quad b: = 1$$

$$a_4: = 2 \quad x_1: = t_1$$

$$h: = 2$$

4. We have a new equation:

$$-3t_1 + 0 ! x_2 + 2x_3 + 2x_4 = 1,$$

$a = \min \{|-3|, |2|, |2|\} = 2$, and

$$i = 3$$

5. $a \dots 1$

$$-3 = 2 ! (-2) + 1$$

$$0 = 2 ! 0 + 0$$

$$2 = 2 ! 1 + 0$$

$$1 = 2 ! 0 + 0$$

8. $x_3 = 2t_1 - 0 ! x_2 - x_4 + 0 - t_2$. Substituting the value of x_3 in the value determined for x_1 we get:

$$x_1 = 2x_2 - 5x_4 + 3t_1 - 2t_2 + 2$$

$$9. \quad a_1: = 1 \quad a_3: = -2$$

$$a_2: = 0 \quad b: = 1$$

$$a_4: = 0 \quad x_3: = t_2$$

$$h: = 3$$

4. We have obtained the equation:

$$1 ! t_2 + 0x_2 - 2t_2 + 0 ! x_4 = 1,$$

$$a = 1, \text{ and}$$

$$i = 1$$

$$6. \quad (A) \quad t_1 = -(0 ! x_2 - 2t_2 + 0 ! x_4 - 1) ! 1 = \\ 2t_2 + 1$$

(B) Substituting the value of t_1 in the values of x_1 and x_3 previously determined, we get:

$$x_1 = 2x_2 - 5x_4 + 4t_2 + 5 \text{ and}$$

$$x_3 = -x_4 + 3t_2 + 2$$

$$(C) \quad x_2: = k_1, \quad x_4: = k_2, \quad t_2 = k_3, \quad k_1, k_2, k_3 \in \mathbb{Z}$$

(D) The general solution of the initial equation is:

$$x_1 = 2k_1 - 5k_2 + 4k_3 + 5$$

$$x_2 = k_1$$

$$x_3 = -k_2 + 3k_3 + 2$$

$$x_4 = k_2$$

k_1, k_2, k_3 are parameters $\in \mathbb{Z}$

Reference

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ANOTHER INTEGER NUMBER ALGORITHM TO SOLVE
 LINEAR EQUATIONS (USING CONGRUENCY)

In the present part a new integer number algorithm for linear equations is presented. This is more "rapid" than W. Sierpinski's presented in [1] in the sense that it reaches the general solution after a smaller number of iterations. Its correctness will be strictly demonstrated.

INTEGER NUMBER ALGORITHM TO SOLVE

LINEAR EQUATIONS

Let us consider the equation (1);
 (the case a_i, b are in \mathbb{Q} ,

$i = 1, n$, is reduced to the case (1) by bringing to the same denominator and eliminating the denominators).
 Let $d = (a_1, \dots, a_n)$. If $d \nmid b$, then the equation does not admit integer solutions, while if $d \mid b$, the equation admits integer solutions (according to a well-known theorem from the theory of numbers).

If the equation accepts solutions and $d \neq 1$, we divide the equation by d . Then, we can agree that $d = 1$ (we do not make any restriction if we consider the maximal co-divisor positive).

(a) Also, if all $a_i = 0$, the equation is trivial; it admits the general integer solution $x_i = k_i \in \mathbb{Z}$, $i = 1, n$ when $b = 0$ (the only case when the general integer solution is n - times undetermined) and does not have solutions when $b \neq 0$.

(b) If i , $1 \leq i \leq n$, so that $a_i = \pm 1$, then the general integer solution is:

$$x_i = -a_i \quad \left(\sum_{\substack{j=1 \\ j \neq i}}^n a_j k_j - b \right) \text{ and } x_s = k_s \in \mathbb{Z}, s \in \{1, \dots, n\} \setminus \{i\}$$

The proof of this assertion was give in [4]. All these cases being trivial, we will leave them aside. The following algorithm can be written:

Input

A linear equation: (2)

$$\sum_{i=1}^n a_i x_i = b, \quad a_i, b \in \mathbb{Z}, \quad a_i \neq 0, \quad i = 1, n$$

with not all $a_i = 0$ and $(a_1, \dots, a_n) = 1$.

Output

The general solution of the equation.

Method

1. $h := 1, p := 1$

2. Calculate $\min_{1 \leq i, j \leq n} \{ |r|, r / a_i \pmod{a_j},$

$r \dots 0, |r| < |a_j|\}$

and determine r and the pair (i, j) for which this minimum can be obtained (when there are more possibilities we have to choose one of them).

3. If $|r| \dots 1$ go on to step 4.

If $|r| = 1$, then

$$x_i := r(-a_j t_h - \sum_{s=1}^n a_s x_s + b) \\ \text{so } \{i, j\}$$

$$x_j := r(a_j t_h + \frac{a_i - r}{a_j} ! - \sum_{s=1}^n a_s x_s + \frac{r - a_i}{a_j} b) \\ \text{so } \{i, j\}$$

(A) Substitute the values thus determined of these unknowns in all the relations (p) , $p = 1, 2 \dots$ (if possible).

(B) From the last relation (p) obtained in
the
algorhythmus substitute in all relations:

$$\overline{(p-1)}, \overline{(p-2)}, \dots, \overline{(1)}.$$

(C) Every relation, starting in order from
 $\overline{(p-1)}$ should be applied the same
procedure as in (B); then $\overline{(p-2)}, \dots, \overline{(3)}$, respectively.

(D) Write the values of the unknowns x_i ,
 $i = 1, n$ from the initial equation writing
the corresponding integer number
parameters
from the right term of these unknowns
with
 k_1, \dots, k_{n-1} , STOP.

4. Write the relation (p):

$$x_j = t_h - \frac{a_i - r}{a_j} x_i;$$

5. Consider $x_j: = t_h \quad h: = h + 1$

$$a_i: = r \quad p: = p + 1$$

The other coefficients and variables remaining
unchanged and go back to step 2.

The Correctness of the Algorithm

Let us consider linear equation (2). Under these conditions, the following properties exist:

Lemma 1. The set $M = \{ |r|, r \equiv a_i \pmod{a_j}, 0 < |r| < |a_j| \}$ has a minimum.

Proof

Obviously $M \neq \emptyset$ and M is finite because the equation has a finite number of coefficients: n and considering all the possible combinations of these, by twos, there is the maximum AR_n^2 (arranged with repetition) = n elements.

Let us show, by reductio ad absurdum, that $M = \emptyset$,

$M = \{ a_i \mid a_i \equiv 0 \pmod{a_j}, \forall i, j \in \{1, \dots, n\} \}$. Hence $a_j =$

$= 0 \pmod{a_i}, \forall i, j \in \{1, \dots, n\}$. Or this is possible only

when $|a_i| = |a_j|, \forall i, j \in \{1, \dots, n\}$, which is equivalent to

$(a_1, \dots, a_n) = a_i, \forall i \in \{1, \dots, n\}$. But $(a_1, \dots, a_n) = 1$

according to a restriction from the assumption. It

follows that $|a_i| = 1, \forall i \in \{1, \dots, n\}$ a fact which contradicts the other restrictions of the assumption.

$M \neq \emptyset$ and finite, it follows that M has a minimum.

Lemma 2. If $|r| = \min_{1 \leq i, j \leq n} M$ then $|r| < |a_i|$, $\forall i \in \overline{0, n}$

Proof

We assume, conversely, that i_0 , $1 \leq i_0 \leq n$, so that $|r| \geq |a_{i_0}|$. Then $|r| \geq \min_{1 \leq j \leq n} \{|a_j|\} = |a_{j_0}| \dots 1$, $1 \leq j_0 \leq n$. Let a_{p_0} , $1 \leq p_0 \leq n$ so that $|a_{p_0}| > |a_{j_0}|$ and a_{p_0} is not divided by a_{j_0} . There is such a coefficient in the equation as $|a_{j_0}|$ is the minimum and not all the coefficients are equal among themselves (conversely, it would mean that $(a_1, \dots, a_n) = a_1 = \pm 1$, which is against the hypothesis) and, again, of the coefficients whose module is greater than $|a_{j_0}|$ not all can be divided by a_{j_0} (conversely, it would similarly result that $(a_1, \dots, a_n) =$

$= a_{j_0} \dots \pm 1$). We write $[a_{p_0} / a_{j_0}] = q_0 \in \mathbb{Z}$ (the whole number part), and $r = a_{p_0} - q_0 a_{j_0} \in \mathbb{Z}$. We have $a_{p_0} / r_0 \pmod{a_{j_0}}$ and $0 < |r_0| < |a_{j_0}| < |a_{i_0}| \leq |r|$. Thus, we have found a r_0 with $|r_0| < |r|$ which contradicts the definition of minimum given to $|r|$. Contrary to the assumption. Thus, $|r| < |a_i|$, $\forall i \in \overline{0, n}$.

Lemma 3. If $|r| = \min M = 1$, for the pair of indices

(i, j), then:

$$x_i = r \left(-a_j t_h - \sum_{\substack{s=1 \\ s \neq \{i, j\}}}^n a_s k_s + b \right)$$

$$x_j = r \left(a_i t_h + \frac{a_i - r}{a_j} \sum_{\substack{s=1 \\ s \neq \{i, j\}}}^n a_s k_s + \frac{r - a_i}{a_j} b \right)$$

$$x_s = k_s \quad 0 \quad Z, \quad s \neq \{1, \dots, n\} \setminus \{i, j\}$$

is the general integer solution of equation (2).

Proof

Let $x_e = x_e^0, e = 1, n$ be a particular integer solution of equation (2). Then $\forall k_s = x_s^0 \quad 0 \quad Z, \quad s \neq \{1, \dots, n\} \setminus \{i, j\}$ and $t_h = x_j^0 + \frac{a_i - r}{a_j} x_i^0 \quad 0 \quad Z$ (because $a_i - r = a_j$), so

that:

$$x_i = r - a_j \left(x_j^0 + \frac{a_i - r}{a_j} x_i^0 \right) - \sum_{\substack{s=1 \\ s \neq \{i, j\}}}^n a_s x_s^0 + b = x_i^0$$

$$x_j = r - a_j \left(x_j^0 + \frac{a_i - r}{a_j} x_i^0 \right) + \frac{a_i - r}{a_j} \sum_{s=1}^{n_0} a_s x_s^0 +$$

$s \in \{i, j\}$

$$+ \frac{r-a_i}{a_j} b = x_j^0$$

and $x_s = k_s = x_s^0, s \in \{1, \dots, n\} \setminus \{i, j\}$.

Lemma 4. Let $|r| \leq 1$ and (i, j) be the pair of indices for which this minimum can be obtained. Again, let the system of linear equation be:

$$(3) \quad a_j t_h + r x_i + \sum_{\substack{s=1 \\ s \neq i, j}}^n a_s x_s = b$$

$$t_h = x_j + \frac{a_i - r}{a_j} x_i$$

Then, $x_e = x_e^0, e = 1, n$ is a particular integer solution for (2) if and only if $x_e = x_e^0, e \in \{1, \dots, n\} \setminus \{j\}$ and

$t_h = t_h^0 = x_j^0 + \frac{a_i - r}{a_j} x_i$ is the particular integer solution

of (3).

Proof

$x_e = x_e^0, e = 1, n$ is a particular integer solution for (2)

$$] \sum_{e=1}^n a_e x_e^0 = b] \sum_{s=1}^n a_s x_s^0 + a_j (x_j^0 + \frac{a_i - r}{a_j} x_i^0) +$$

s...{i, j}

$$+ rx_i^0 = b$$

$$] a_j t_h^0 + rx_i^0 + \sum_{s=1}^n a_s x_s^0 = b \text{ and } t_h^0 = x_j^0 +$$

s...{i, j}

$$+ \frac{a_i - r}{a_j} x_i^0 \in \mathbb{Z}$$

$$] x_e^0 = x_e, e \in \{1, 2, \dots, n\} \setminus \{j\}$$

and $t_h^0 = t_h$ is a particular integer solution for (3).

Lemma 5. The former algorithm is finite.

Proof

When $|r| = 1$ the algorithm stops at step 3. We will discuss the case when $|r| \geq 1$. According to the definition of r , $|r| \geq 1$. We show that the row of $r - s$ successively obtained by following the algorithm several times is strictly decreasing to 1. Let r_1 be the first obtained by following the algorithm one time. $|r_1| \geq 1$, go on to steps 4 and, then 5. According to lemma 2 $|r_1| <$

$< |a_i|$, $v_i = 1, n$. Now we shall follow the algorithm a second time, but this time for an equation in which r_1 (according to step 5) is substituted for a_i . Again, according to lemma 2, the new $|r|$ written $|r_2|$ will have the property: $|r_2| < |r_1|$. We will get to $|r| = 1$ as $|r| \geq 1$ and $|r| < 4$, and if $|r| \geq 1$, following the algorithm once again we get $|r| < |r_1|$, a.s.o. Hence, the algorithm has a finite number of repetitions.

Theorem of Correctness. The former algorithm calculates correctly the general integer solution of the linear equation (2).

Proof

According to lemma 5 the algorithm is finite. From lemma 1 it follows that the set M has a minimum, hence step 2 of the algorithm has meaning. When $|r| = 1$ it was shown in lemma 3 that step 3 of the algorithm calculates correctly the general integer solution of the respective equation (the equation that appears at step 3). In lemma 4 it is shown that if $|r| \geq 1$, by the substitutions steps 4 and 5 introduce in the initial equation the general integer solution remains unchanged. That is, we pass from the initial equation to a linear system having the same

general solution as the initial equation. The variable h is a counter of the newly introduced variables which are used to successively decompose the system in systems of two linear equations. The variable p is a counter of the substitutions of variables (the relations, at a given moment, between certain variables).

When the initial equation was decomposed to $|r| = 1$, we have to follow the reverse way: i.e., to compose its general integer solution. This reverse way is directed by the substeps 3(A), 3(B) and 3(C). The substep 3(D) has only an aesthetic role, i.e., to have the general solution

under the form: $x_i = f_i(k_1, \dots, k_{n-1})$, $i = 1, n$, f_i being linear functions with integer number coefficients. This "if possible" shows that substitutions are not always possible. But when they are we have to make all the possible substitutions.

Note 1. The former algorithm is written under a form easy to introduce in the computer.

Note 2. The former algorithm is more "rapid" than that of W. Sierpinski's 1, i.e., the general integer solution is reached after a smaller number of iterations (or, at least, the same) for any linear equation (2). In the first place, both aim at obtaining the coefficient ± 1 for at least one unknown variable. While Sierpinski

started only by chance by decomposing the greatest coefficient in the module (writing it under the form of a sum between a multiple of the following smaller coefficient (in the module) and the rest), in our algorithm this decomposition is not accidental but always seeks the smallest $|r|$ and also chooses the coefficients a_i and a_j for which this minimum is achieved. That is, we test from the beginning the shortest way to the general integer solution. Sierpinski does not attempt at finding the shortest way; he knows that his way will take him to the general integer solution of the equation and is not interested in how long it will be. However, when an algorithm is introduced in a computer, it is preferable that the computer time should be as short as possible.

Example 1

Let us solve in \mathbb{Z}^3 the equation: $17x - 7y + 10z = -12$. We apply the former algorithm.

1. $h = 1, p = 1$
2. $r = 3, i = 3, j = 2$
3. $|3| \dots 1, \text{ go on to step 4.}$
4. (1)

$$y = t_1 = \frac{10-3}{-7} ! z = t_1 + z$$

5. Consider $y: = t_1$ $h: = 2$

$a_3: = 3$ $p: = 2$

the other coefficients and variables remaining unchanged, go back to step 2.

2. $r = -1, i = 1, j = 3$

3. $|-1| = 1$

$$x = -1(-3t_2 - (-7t_1) - 12) = 3t_2 - 7t_1 - 12$$

$$z = -1(17t_2 + (-7t_1) ! \frac{17 - (-1)}{3} + \frac{-1 - 17}{3} (-12)) = \\ = -17t_2 + 42t_1 - 72$$

(A) We substitute the values of x and z thus determined in the only relation (p) we have:

$$(1) \quad y = t_1 + z = -17t_2 + 43t_1 - 72$$

(B) The substitution is not possible.

(C) The substitution is not possible.

(D) The general integer solution of the equation is:

$$x = 3k_1 - 7k_2 + 12$$

$$y = -17k_1 + 43k_2 - 72$$

$$z = -17k_1 + 42k_2 - 72; \quad k_1, k_2 \in \mathbb{Z}$$

References

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INTEGER NUMBER SOLUTIONS OF LINEAR SYSTEMS

Definitions and Properties of the
Integer Solution of a Linear System

$$\text{Let } \sum_{j=1}^n a_{ij}x_j = b_i, \quad i = 1, m \quad (1)$$

A linear system with all its coefficients' integer numbers (the case with rational coefficients is reduced to the same).

Definition 1. $x_j = x_j^0, \quad j = 1, n$ is a particular integer solution of (1) if $x_j^0 \in \mathbb{Z}, \quad j = 1, n$ and $\sum_{j=1}^n a_{ij}x_j^0 = b_i, \quad i = 1, m$. Let the functions be $f_j : \mathbb{Z}^h \rightarrow \mathbb{Z}, \quad j = 1, n$, where $h \in \mathbb{N}^*$.

Definition 2. $x_j = f_j(k_1, \dots, k_h), \quad j = 1, n$, is the general integer solution for (1) if:

$$(a) \quad \sum_{j=1}^n a_{ij}f_j(k_1, \dots, k_h) = b_i, \quad i = 1, m$$

of $(k_1, \dots, k_h) \in \mathbb{Z}^h$;

(b) For any $x_j = x_j^0, \quad j = 1, n$, particular integer solution of (1), there is $(k_1, \dots, k_h) \in \mathbb{Z}^h$ so that $f_j(k_1, \dots, k_h) = x_j^0, \quad j = 1, n$.

$$0 \in \mathbb{Z} \text{ so that } f_j(k_1, \dots, k_h) = x_j^0, \quad j = 1, n.$$

(In other words, the general solution is the solution that comprises all the other solutions.)

*

Propriety 1

A general solution of a linear system of m equations with n unknowns, $r(A) = m < n$ is $n - m$ times undetermined.

Proof

We assume by reductio ad absurdum that it is of order r , $1 \leq r < n - m$ (the case $r = 0$, i.e., the solution is particular, is trivial). It follows that the general solution is of the form:

$$\begin{aligned}
 x_1 &= u_{11}p_1 + \dots + u_{1r}p_r + v_1 \\
 (S_1) \quad &\cdot \\
 &\cdot \\
 &\cdot \\
 x_n &= u_{n1}p_1 + \dots + u_{nr}p_r + v_n, \quad u_{ih}, \quad v_i \in \mathbb{Z}
 \end{aligned}$$

p_h = parameters $\in \mathbb{Z}$

We prove that there are $n - m$ times undetermined solutions. The homogenous linear system (1), solved in r admits the solution:

$$\begin{aligned}
 x_1 &= \frac{1}{D_{m+1}} x_{m+1} + \dots + \frac{1}{D_n} x_n \\
 &\quad \cdot \quad \cdot \\
 &\quad \cdot \\
 &\quad \cdot \\
 x_m &= \frac{m}{D_{m+1}} x_{m+1} + \dots + \frac{m}{D_n} x_n
 \end{aligned}$$

Let $x_i = x_i^0$, $i = 1, n$ be a particular solution of the linear system (1).

Considering

$x_{m+1} = D^{-1} k_{m+1}$, we get a solution

$$\begin{aligned}
 &\cdot \\
 &\cdot \quad x_1 = \frac{1}{D_{m+1}} k_{m+1} + \dots + \frac{1}{D_n} k_n + x_1^0 \\
 &\cdot \\
 &x_n = D^{-1} k_n \quad \cdot \\
 &\cdot \\
 &\cdot \quad x_m = \frac{m}{D_{m+1}} k_{m+1} + \dots + \frac{m}{D_n} k_n + x_m^0 \\
 &\cdot \quad x_{m+1} = D^{-1} k_{m+1} + x_{m+1}^0 \\
 &\cdot \\
 &\cdot \\
 &x_n = D^{-1} k_n + x_n^0, \quad k_j = \\
 &\quad = \text{parameters } 0 \text{ Z}
 \end{aligned}$$

which depends on the $n - m$ independent parameters, for the system (1). Let the solution be $n - m$ times undetermined:

$$\begin{aligned}
 x_1 &= c_{11}k_1 + \dots + c_{1n-m}k_{n-m} + d_1 \\
 \cdot & \\
 \cdot & \\
 x_n &= c_{n1}k_1 + \dots + c_{nn-m}k_{n-m} + d_n
 \end{aligned}$$

$c_{ij}, d_i \in \mathbb{Z}$, k_j = parameters $\in \mathbb{Z}$

(There are such solutions, we have proven it before.)

Let the system be:

$$\begin{aligned}
 a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 a_{m1}x_1 + \dots + a_{mn}x_n &= b_m \quad \begin{matrix} a_{ij}, b_i \in \mathbb{Z} \\ x_i = \text{unknowns} \in \mathbb{Z} \end{matrix}
 \end{aligned}$$

I. The case $b_i = 0, i = 1, m$ results in a homogenous
linear system:

$$\begin{aligned}
 a_{i1}x_i + \dots + a_{in}x_n &= 0, \quad i = 1, m \\
 \begin{aligned}
 (S_2) \quad Y \quad a_{i1}(c_{i1}k_1 + \dots + c_{1n-m}k_{n-m} + d_1) + \dots + a_{in}(c_{n1}k_1 + \\
 + \dots + c_{nn-m}k_{n-m} + d_n) &= 0
 \end{aligned}
 \end{aligned}$$

$$0 = (a_{i1}c_{11} + \dots + a_{in}c_{n1})k_1 + \dots + (a_{i1}c_{1n-m} + \dots + a_{in}c_{n-n-m})k_{n-m} + (a_{i1}d_1 + \dots + a_{in}d_n), \quad \forall k_j \in \mathbb{Z}$$

For $k_1 = \dots = k_{n-m} = 0$ $\Rightarrow a_{i1}d_1 + \dots + a_{in}d_n = 0$

For $k_1 = \dots = k_{h-1} = k_{h+1} = \dots = k_{n-m} = 0$ and $k_h = 1 \Rightarrow (a_{i1}c_{ih} + \dots + a_{in}c_{nh}) + (a_{i1}d_1 + \dots + a_{in}d_n) = 0 \Rightarrow a_{i1}c_{1h} + \dots + a_{in}c_{nh} = 0$

$\forall i = 1, m, \forall h = 1, n-m$

Vect. $V_h = \begin{bmatrix} c_{1h} \\ \vdots \\ c_{nh} \end{bmatrix}, \quad h = 1, n-m, \quad$ are the particular solutions of the system.

$V_h, \quad h = 1, n-m$ are also linearly independent because the solution is $n - m$ times undetermined. $\{V_1, \dots, V_{n-m}\} + d$ is a linear variety that includes the solutions of the system obtained from (S_2) . Similarly, for (S_1) we deduce

that $U_{1s}, \dots, U_{rs}, \quad s = 1, r$ are particular solutions of

U_{ns}

the given system and are linearly independent because

$(S_1) = r -$ times undetermined solution and

$$V = \begin{matrix} v_1 \\ \vdots \\ v_r \\ \vdots \\ v_n \end{matrix} = a \text{ solution of the given system.}$$

The case (a). U_1, \dots, U_r, ∞ = linearly dependent, it follows that $\{U_1, \dots, U_r\}$ is a free submodule of order $r < n - m$ of solutions of the given system, then, it follows that there are solutions that belong to $\{V_1, \dots, V_{n-m}\} + d$ and which do not belong to $\{U_1, \dots, U_r\}$, a fact which contradicts the assumption that (S_1) is the general solution.

The case (b). U_1, \dots, U_r, V = linearly independent. $\{U_1, \dots, U_r\} + V$ is a linear variety that comprises the solutions of the given system, which were obtained from (S_1) . It follows that the solution belongs to $\{V_1, \dots, V_{n-m}\} + d$ and does not belong to $\{U_1, \dots, U_r\} + V$, a fact which is in contradiction with the assumption that (S_1) is the general solution.

II. When there is an $i \neq 1, m, b_i \neq 0$ Y nonhomogeneous

linear system

$$a_{i1}x_1 + \dots + a_{in}x_n = b_i, \quad i = 1, m$$

$$(S_2) \quad \begin{aligned} & Y a_{11} (c_{11}k_1 + \dots + c_{1n-m}k_{n-m} + d_1) + \dots + \\ & + a_{1n} (c_{n1}k_1 + \dots + c_{nn-m}k_{n-m} + d_n) = b_1; \end{aligned}$$

it follows that

$$\begin{aligned}
 & Y (a_{i1}c_{11} + \dots + a_{in}c_{n1}) k_1 + \dots + (a_{i1}c_{1n-m} + \\
 & + \dots + a_{in}c_{n-m}) k_{n-m} + (a_{i1}d_1 + \dots + a_{in}d_n) = \\
 & = b_i;
 \end{aligned}$$

for $k_1 = \dots = k_{n-m} = 0$ $Y a_{i1}d_1 + \dots + a_{in}d_n = b_i;$

for $k_1 = \dots = k_{j-1} = k_{j+1} = \dots = k_{n-m} = 0$ and $k_j = 1$ Y

$$Y (a_{i1}c_{1j} + \dots + a_{in}c_{nj} + (a_{ind_1} + \dots + a_{1n} d_n) = b_i$$
 it follows that

$$a_{i1}c_{1j} + \dots + a_{in}c_{nj} = 0$$

$$a_{i1}d_1 + \dots + a_{in}d_n = b_i, \quad \forall i = 1, m, \quad \forall j = 1, n-m;$$

\mathbf{C}_{1j}

$V_j = \begin{matrix} \cdot \\ \cdot \\ c_{nj} \end{matrix}, j = 1, n-m$ are linearly independent
 because the solution (S_2) in
 $n - m$ times is undetermined.

?! $V_j, j = 1, n-m$ and $d = \begin{matrix} d_1 \\ \cdot \\ d_n \end{matrix}$ are linearly independent.

We assume that they are not linearly independent. It follows that

$$d = s_1 V_1 + \dots + s_{n-m} V_{n-m} =$$

$$\begin{matrix} s_1 c_{11} + \dots + s_{n-m} c_{1n-m} \\ \cdot \\ \cdot \\ \cdot \\ s_1 c_{n1} + \dots + s_{n-m} c_{nn-m} \end{matrix}$$

Irrespective of $i = 1, m$:

$$\begin{aligned} b_1 &= a_{i1} d_1 + \dots + a_{in} d_n = a_{i1} (s_1 c_{11} + \dots + s_{n-m} c_{1n-m}) + \\ &+ \dots + a_{in} (s_1 c_{n1} + \dots + s_{n-m} c_{nn-m}) = (a_{i1} c^{11} + \dots + \\ &+ a_{in} c_{n1}) s_1 + \dots + (a_{i1} c_{1n-m} + \dots + a_{in} c_{nn-m}), s_{n-m} = \\ &0 \end{aligned}$$

Then, $b_i = 0$, irrespective of $i = 1, m$, contradicts the

hypothesis (that if there is an $i \in 1, m$, $b_i \neq 0$). It follows that v_1, \dots, v_{n-m}, d are linearly independent.

$\{V_1, \dots, V_{n-m}\} + d$ is a linear variety that contains the solutions of the nonhomogeneous system, solutions obtained from (S_2) . Similarly, from (S_1) it follows that $\{G_1, \dots, G_r\} + V$ is a linear variety containing the solutions of the nonhomogeneous system, obtained from (S_1) .

$n-m > r$ follows that there are solutions of the system that belong to $\{V_1, \dots, V_{n-m}\} + d$ and which do not belong to $\{G_1, \dots, G_r\} + V$ (it contradicts the fact that (S_1) is the general solution). Then, it results that the general solution depends on the $n-m$ independent parameters.

Theorem 1. The general solution of a nonhomogeneous linear system is equal to the general solution of an associated linear system plus a particular solution of the nonhomogeneous system.

Proof

Let the homogeneous linear solution:

$$\begin{aligned}
 a_{11}x_1 + \dots + a_{1n}x_n &= 0 \\
 \cdot & \\
 \cdot & \quad , \quad (AX = 0) \\
 \cdot \\
 a_{m1}x_1 + \dots + a_{mn}x_n &= 0
 \end{aligned}$$

with the general solutions:

$$\begin{aligned} x_1 &= c_{11}k_1 + \dots + c_{1n-m}k_{n-m} + d_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n &= c_{n1}k_1 + \dots + c_{nn-m}k_{n-m} + d_n \end{aligned}$$

and $\begin{aligned} x_1 &= x_1^0 \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n &= x_n^0 \end{aligned}$

a particular solution of the nonhomogeneous linear system
 $AX = b$;

$$\begin{aligned} ?! \quad x_1 &= c_{11}k_1 + \dots + c_{1n-m}k_{n-m} + d + x_1^0 \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n &= c_{n1}k_1 + \dots + c_{nn-m}k_{n-m} + d_n + x_n^0 \end{aligned}$$

is a solution of the nonhomogeneous linear system.

We have written

$$A = \begin{matrix} a_{11} & \dots & a_{1n} & x_1 & b_1 & 0 \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ a_{m1} & \dots & a_{mn} & x_n & b_m & 0 \end{matrix}, \quad X = \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}, \quad b = \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}, \quad 0 = \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix}$$

(vector of dimension m),

$$\begin{array}{ccccccc}
 & 0 & & & & & \\
 & k_1 & & c_{11} & \dots & c_{1n-m} & d_1 \\
 x_1 & & & & & & \\
 & \cdot & & \cdot & & & \cdot \\
 & & & & & & \\
 K = & \cdot & , & C = & \cdot & , & d = \cdot , x^0 = \cdot \\
 ; & & & & & & \\
 & \cdot & & \cdot & & & \cdot \\
 & 0 & & c_{n1} & \dots & c_{nn-m} & d_n \\
 x_n & & & & & &
 \end{array}$$

$$AX = A(Ck + d + x^0) = A(Ck + d) + AX^0 = b + 0 = b$$

$$\begin{array}{c}
 \text{We will prove that irrespective of } x_1 = y_1^0 \\
 \cdot \\
 \cdot \\
 \cdot \\
 x_n = y_n^0
 \end{array}$$

there is a particular solution of the nonhomogeneous system

$$\begin{array}{ll}
 k_1 = k_1^0 & 0 \quad Z \\
 \cdot & \\
 \cdot & \\
 \cdot & 0 \\
 k_{n-m} = k_{n-m}^0 & 0 \quad Z
 \end{array}
 \quad \text{with the property:}$$

$$\begin{array}{ll}
 x_1 = c_{11}k_1^0 + \dots + c_{1n-m}k_{n-m}^0 + d_1^0 & x_1 = y_1^0 \\
 \cdot & \\
 \cdot & \\
 \cdot & 0 \\
 x_n = c_{n1}k_1^0 + \dots + c_{nn-m}k_{n-m}^0 + d_n^0 & x_n = y_n^0
 \end{array}$$

We write $Y^0 = \begin{matrix} 0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ y_n \end{matrix}$

We demonstrate that those $\begin{matrix} 0 \\ k_j \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ z \end{matrix}$, $j = 1, n-m$ are those for

which $A(CX^0 + d) = 0$ (there are such $\begin{matrix} 0 \\ x_j \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ z \end{matrix}$ because

$$\begin{matrix} x_1 = 0 \\ \cdot \\ \cdot \\ \cdot \\ x_n = 0 \end{matrix}$$

is a particular solution of the homogenous linear system
and $X = CK + d$ is a general solution of the
nonhomogeneous

linear system) $A(CK^0 + d + X^0 - Y^0) = A(CK^0 + d) + AX^0 -$
 $- AY^0 = 0 + b - b = 0$.

Propriety 2. The general solution of a homogenous
linear system can be written under the form:

$$\begin{matrix} x_1 = c_{11}k_1 + \dots + c_{1n-m}k_{n-m} \\ \cdot \\ \cdot \\ x_n = c_{n1}k_1 + \dots + c_{nn-m}k_{n-m} \end{matrix},$$

(SG)
(2)

k_j = a parameter belonging to Z (with $d_1 = d_2 = \dots = d_n = 0$)

Proof

(SG) = general solution. It results that (SG) is $(n-m)$ times undetermined.

Let (SG) of the form be

$$\begin{aligned} x_1 &= c_{11}p_1 + \dots + c_{1n-m}p_{n-m} + d_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n &= c_{n1}p_1 + \dots + c_{nn-m}p_{n-m} + d_n \end{aligned}$$

with not all $d_i = 0$; we demonstrate that it can be written under the form (2); the system admits the trivial solution

$$\begin{aligned} x_1 &= 0 \ 0 \ Z \\ &\cdot \\ &\cdot \quad ; \\ &\cdot \\ x_n &= 0 \ 0 \ Z \end{aligned}$$

it results that there are $p_j \neq 0$, $j = 1, n-m$ -----

$$x_1 = c_{11}^0 p_1 + \dots + c_{1n-m}^0 p_{n-m} + d_1 = 0$$

(4)

$$\begin{aligned} & \cdot \\ & \cdot \\ x_n &= c_{n1} \overset{0}{p_1} + \dots + c_{nn-m} \overset{0}{p_{n-m}} + d_n = 0 \end{aligned}$$

Substituting $p_j = k_j + \overset{0}{p_j}$, $j = 1, n-m$ in (3)

$$\begin{aligned} k_j & 0 Z \\ \overset{0}{p_j} & 0 Z A Y \overset{0}{p_j} 0 Z \end{aligned}$$

$$\begin{aligned} p_j & 0 Z \\ \overset{0}{p_j} & 0 Z A Y k_j = p_j - \overset{0}{p_j} 0 Z \end{aligned}$$

which means that they do not make any restrictions.

It results that

$$\begin{aligned} x_1 &= c_{11}k_1 + \dots + c_{1n-m}k_{n-m} + (c_{11} \overset{0}{p_1} + \dots + c_{1n-m} \overset{0}{p_{n-m}} + d_1) \\ & \cdot \\ & \cdot \\ x_n &= c_{n1}k_1 + \dots + c_{nn-m}k_{n-m} + (c_{n1} \overset{0}{p_1} + \dots + c_{nn-m} \overset{0}{p_{n-m}} + d_n) \end{aligned}$$

But $c_{h1} \overset{0}{p_1} + \dots + c_{hn-m} \overset{0}{p_{n-m}} + d_h = 0$, $h = 1, n$ (from (4)).

Then the general solution is of the form:

$$\begin{aligned} x_1 &= c_{11}k_1 + \dots + c_{1n-m}k_{n-m} \\ & \cdot \\ & \cdot \end{aligned}$$

$$\mathbf{x}_n = c_{n1}k_1 + \dots + c_{n,n-m}k_{n-m}$$

k_j = parameters 0 Z, $j = 1, n-m$; it results that $d_1 = d_2 = \dots = d_n = 0$

Theorem 2. Let the homogenous linear system be:

$$a_{11}x_1 + \dots + a_{1n}x_n = 0$$

.

.

.

$$a_{m1}x_1 + \dots + a_{mn}x_n = 0, \quad r(A) = m;$$

$(a_{h1}, \dots, a_{hn}) = 1, \quad h = 1, m$ and the general solution

$$x_1 = c_{11}k_1 + \dots + c_{1,n-m}k_{n-m}$$

.

.

.

$$x_n = c_{n1}k_1 + \dots + c_{n,n-m}k_{n-m}$$

then $(a_{h1}, \dots, a_{hi-1}, a_{hi+1}, \dots, a_{hn}) \mid (c_{i1}, \dots, c_{in-m})$

irrespective of $h = 1, m$ and $i = 1, n$.

Proof

Let some arbitrary be $h \in 1, m$ and some arbitrary $i \in 0$

$1, n$; $a_{h1}x_1 + \dots + a_{hi-1}x_{i-1} + a_{hi+1}x_{i+1} + \dots + a_{hn}x_n = a_{hi}x_i$.

Because $(a_{h1}, \dots, a_{hi-1}, a_{hi+1}, \dots, a_{hn}) \mid a_{hi}$ it results that $d = (a_{h1}, \dots, a_{hi-1}, a_{hi+1}, \dots, a_{hn}) \mid x_i$, irrespective of the value of x_i in the vector of particular solutions; for $k_2 = k_3 = \dots = k_{n-m} = 0$ and $k_1 = 1$ we get the particular solution:

$$\begin{aligned} x_1 &= c_{11} \\ &\cdot \\ &\cdot \\ &\cdot \\ x_i &= c_{i1} \text{ Y } d \mid c_{i1} \text{ and so on} \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n &= c_{n1} \end{aligned}$$

for $x_1 = k_2 = \dots = k_{n-m-1} = 0$ and $k_{n-m} = 1$ the following particular solution results:

$$\begin{aligned} x_1 &= c_{1n-m} \\ &\cdot \\ &\cdot \\ &\cdot \\ x_i &= c_{in-m} \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n &= c_{nn-m} \end{aligned}$$

it results that $d \mid c_{in-m}$; hence, $d \mid c_{ij}$, $j = 1, n-m$ Y $\overbrace{\quad \quad \quad}^{\text{-----}}$ $d \mid (c_{i1}, \dots, c_{in-m})$.

Theorem 3.

$x_1 = c_{11}k_1 + \dots + c_{1n-m}k_{n-m}$ $c_{ij} \in \mathbb{Z}$ being given
 If
 .
 .
 .
 $x_n = c_{n1}k_1 + \dots + c_{nn-m}k_{n-m}$, $k_j = \text{parameters } \in \mathbb{Z}$

is the general solution of the homogenous linear system

$a_{11}x_1 + \dots + a_{1n}x_n = 0$ $m < n$;
 .
 .
 .
 $a_{m1}x_1 + \dots + a_{mn}x_n = 0$ $r(A) = m$;

 then $(c_{1j}, \dots, c_{nj}) = 1$, $\forall j = 1, n-m$.

Proof

We assume, by reductio ad absurdum, that there is

 $j_0 \in 1, n-m$: $(c_{1j_0}, \dots, c_{nj_0}) = d \dots 1$, we consider the
 maximal co-divisor > 0 ; we reduce the case when the
 maximal co-divisor is $-d$ to the case when it is equal to d
 (nonrestrictive hypothesis); then the general solution can
 be written under the form:

$$x_1 = c_{11}k_1 + \dots + c_{1j_0} \frac{dk_{j_0}}{d} + \dots + c_{1n-m}k_{n-m}$$

.

(5)

$$x_n = c_{n1}k_1 + \dots + c_{nj_0}dk_{j_0} + \dots + c_{nn-m}k_{n-m}$$

where $d = (c_{ij_0}, \dots, c_{nj_0})$, $c_{ij_0} = d \cdot c_{ij_0}$ and $(c_{ij_0}, \dots, c_{nj_0}) = 1$.

We prove that

$$\begin{aligned} x_1 &= c_{1j_0} \\ &\vdots \\ &\vdots \\ x_n &= c_{nj_0} \end{aligned}$$

is a particular solution of the homogenous linear system.

We write

$$C = \begin{matrix} c_{11} & \dots & c_{ij_0} & d & \dots & c_{1n-m} \\ \vdots & & \vdots & & & \vdots \\ c_{n1} & \dots & c_{nj_0} & d & \dots & c_{nn-m} \end{matrix} \quad k = \begin{matrix} k_1 \\ \vdots \\ k_{j_0} \\ \vdots \\ k_{n-m} \end{matrix};$$

$x = c \cdot k$ the general solution;

$$a_{11} \dots a_{1n}$$

We know $AX = 0$ $Y A(CK) = 0$, $A = \begin{matrix} & & & \\ & \ddots & & \\ & & a_{11} & \dots & a_{mn} \end{matrix}$;

We assume that the principal variants are x_1, \dots, x_m (if not we have to renumber). It follows that x_{m+1}, \dots, x_n is the secondary variant.

For $k_1 = \dots = k_{j_0-1} = k_{j_0+1} = \dots = k_{n-m} = 0$ and $k_{j_0} = 1$ we get a particular solution of the system

$$\begin{matrix} x_1 = c_{1j_0} & d & & c_{1j} & d & & c_{1j_0} \\ \cdot & & & \cdot & 0 & & \cdot \\ \cdot & & Y & 0 = A & \cdot & = d & ! & A & \cdot & Y \\ \cdot & & & & \cdot & & & & \cdot & \\ x_n = c_{nj_0} & d & & c_{nj_0} & d & & c_{nj_0} \\ & 0 & & & & & & \end{matrix}$$

$$\begin{matrix} & c_{1j_0} & & x_1 = c_{1j} & & \\ & \cdot & & \cdot & 0 & \\ Y & A & \cdot & = 0 & Y & \cdot \\ & & & & & \cdot \\ & c_{nj_0} & & x_n = c_{nj_0} & & \end{matrix}$$

is the particular solution of the system.

We demonstrate that this particular solution cannot be obtained by

$$(6) \quad \begin{matrix} x_1 = c_{11}k_1 + \dots & c_{1j} & dk_{j_0} & + \dots + c_{1n-m}k_{n-m} = c_{1j_0} \\ \cdot & & 0 & \\ & \cdot & & \end{matrix}$$

$$\begin{aligned}
 x_n &= c_{n1}k_1 + \dots + c_{nj_0}dk_{j_0} + \dots + c_{nn-m}k_{n-m} = c_{nj_0} \\
 \\
 x_{m+1} &= c_{m+1}k_1 + \dots + c_{m+1}dk_{j_0} + \dots + c_{m+1,n-m}k_{n-m} = c_{m+1} \\
 &\dots \\
 (7) \quad &\dots \\
 x_n &= c_{n1}k_1 + \dots + c_{nj_0}dk_{j_0} + \dots + c_{nn-m}k_{n-m} = c_{nj_0} \\
 \\
 &\begin{array}{ccccccc}
 & c_{m+1,1} & \dots & c_{m+1,j_0} & \dots & c_{m+1,n-m} \\
 & \cdot & & \cdot & 0 & \cdot & \\
 & \cdot & & \cdot & & \cdot & \\
 & \cdot & & \cdot & & \cdot & \\
 & c_{11} & \dots & c_{nj_0} & \dots & c_{n,n-m} & = 1 \\
 & \dots & & \dots & & \dots & \\
 & c_{n1} & \dots & c_{nj_0} & d & \dots & c_{n,n-m} \\
 & & & & \cdot & & \\
 & & & & \cdot & & \\
 & & & & \cdot & & \\
 & & & & c_{n1} & \dots & c_{nj_0} & d & \dots & c_{n,n-m}
 \end{array} \\
 &\text{because } d \neq 0
 \end{aligned}$$

It is important to point out the fact that those $k_j = k_{j_0}^0$,

$j = 1, n-m$ that satisfy system (7) also satisfy system (6),

because, otherwise (6) would not satisfy the definition of the solution of a linear system of equations (i.e., considering system (7) the hypothesis was not restrictive). From $X_j = 0$ it follows that (6) is not the general solution of the homogenous linear system

contrary to the hypothesis); then $(c_{1j} \dots c_{nj}) = 1$,

irrespective of $j = 1, n-m$.

Propriety 3. Let the linear system be

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

.

.

.

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

$a_{ij}, b_i \in \mathbb{Z}$, $r(A) = m < n$, x_j = unknowns $\in \mathbb{Z}$

Solved in R, we get

$$x_1 = f_1(x_{m+1}, \dots, x_n)$$

.

;

x_1, \dots, x_m are the

.

$$x_m = f_m(x_{m+1}, \dots, x_n)$$

main variants where f_i are linear functions of the form:

$$f_i = \frac{c_{m+1}x_{m+1} + \dots + c_nx_n + e_i}{d_i} \text{ where } c_{m+j}, d_i, e_i \in \mathbb{Z}; i = 1, m; \text{ ---}$$

 $j = 1, n-m$.

If $\frac{e_i}{d_i} \in \mathbb{Z}$ irrespective of $i = 1, m$ then the linear system

d_i

admits integer solution.

Proof

For $1 \leq i \leq m$, $x_i \in \mathbb{Z}$, then $f_j \in \mathbb{Z}$. Let:

$$x_{m+1} = u_{m+1}k_{m+1}$$

.

.

.

$$x_n = u_n k_n$$

Solution

.

.

$$x_1 = v_{m+1}^{1/m+1} k_{m+1} + \dots + v_n^{1/n} k_n + \frac{e_1}{d_1}$$

.

.

$$x_m = v_{m+1}^{m/m+1} k_{m+1} + \dots + v_n^{m/n} k_n + \frac{e_m}{d_m}$$

where u_{m+1} is the maximal co-divisor of the denominators of

the fractions $\frac{c_{m+j}}{d_i}$, $i = 1, m$, $j = 1, n-m$ calculated after their simplification when they were irreducible.

$$v_{m+j}^i = \frac{c_{m+j} u_{m+1}}{d_i} \in \mathbb{Z}; \text{ this is a solution } n-m \text{ times}$$

undetermined (depends on $n-m$ independent parameters:

k_{m+1}, \dots, k_n) but is not a general solution.

if Propriety 4. Under the conditions of property 3,

there is an $i_0 \in \{0, 1, \dots, m\}$: $f_{i_0} = u_{m+1}x_{m+1} + \dots + u_n x_n + \frac{e_{i_0}}{d_{i_0}}$

with $u_{m+j} \neq 0 \in \mathbb{Z}$, $j = 1, n-m$ and $\dots \neq 0 \in \mathbb{Z}$ then the system does not admit integer solution.

Proof

if $x_{m+1}, \dots, x_n \in \mathbb{Z}$ it results in $x_{i_0} \neq \mathbb{Z}$.

Theorem 4. Let the linear system be

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

.

.

.

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

$a_{ij}, b_i \in \mathbb{Z}$, x_j = unknowns $\in \mathbb{Z}$, $r(A) = m < n$. If there are

indices $1 \leq i_1 < \dots < i_m \leq n$, $i_h \in \{1, 2, \dots, n\}$,

with the property:

$$\begin{matrix} a_{1i_1} & \dots & a_{1i_m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{mi_1} & \dots & a_{mi_m} \end{matrix} \quad \text{and}$$

$$\begin{array}{ccccccc}
 \frac{a_x}{i_1} & = & b_1 & a_{1i_2} & \dots & a_{1i_m} & \\
 \cdot & & \cdot & \cdot & & \cdot & \text{is divided by} \\
 \cdot & & \cdot & \cdot & & \cdot & \\
 \cdot & & \cdot & \cdot & & \cdot & \\
 \cdot & & b_m & a_{mi_2} & \dots & a_{mi_m} & \\
 \cdot & & & & & & \\
 \cdot & & & & & & \\
 \cdot & & & & & & \\
 \cdot & & a_{1i_1} & \dots & a_{1i_{m-1}} & b_1 & \\
 \frac{a_x}{i_m} & = & \cdot & & \cdot & & \text{is divided by} \\
 \cdot & & \cdot & & \cdot & & \\
 \cdot & & \cdot & & \cdot & & \\
 a_{mi_1} & \dots & a_{mi_{m-1}} & b_m & & &
 \end{array}$$

then the system admits integer number solutions.

Proof

We use property 3

$$\begin{array}{ccccccc}
 d_i & = & \frac{a}{i}, & i & = & 1, m; & e_{i_h} = \frac{a_x}{i_h}, & h & = & 1, m
 \end{array}$$

Note 1. Conversely, it is not true.

Consequence 1. Any homogenous linear system admits integer number solutions (beside the trivial one); $r(A) = m < n$.

Proof

$\frac{a_x}{i_h} = 0 \neq \frac{a}{i_h}$, irrespective of $h = 1, m$.

Consequence 2. If $\frac{a}{i_h} = \pm 1$, it follows that the linear system admits integer number solutions.

Proof

$\frac{a_x}{i_h} \neq \pm 1$, irrespective of $h = 1, m$;

$\frac{a_x}{i_h} \neq 0 \in \mathbb{Z}$.

FIVE INTEGER NUMBER ALGORITHMS TO
SOLVE LINEAR SYSTEMS

This chapter further extends the results obtained in 4 and 5 (from linear equations to linear systems). Each algorithm is strictly demonstrated and then an example is given.

Five integer number algorithms to solve linear systems are further given.

Algorithm 1 (method of substitution)

(Although simple, this algorithm requires complex calculus but is, nevertheless, advantageous in introducing it in the computer).

Some integer number equations are initially solved (which is usually simpler) by means of one of the algorithms 4 or 5. (If there is an equation of the system which does not admit integer systems, then the integer system does not admit integer systems. Stop.) the general integer solution of the equation will depend on $n-1$ integer number parameters (see 5):

$$(p_1) \quad x_{i_1} = f_{i_1}^{(1)} (k_1, \dots, k_{n-1}^{(1)}), \quad i = 1, n \text{ where all}$$

the functions $f_{i_1}^{(1)}$ are linear with integer number coefficients.

This general integer number system (p_1) is introduced in the other $m-1$ equations of the system. We get a new system of $m-1$ equations with $n-1$ unknown variables:

$k_{i_1}^{(1)}$, $i_1 = 1, n-1$, which is also to be solved in integer numbers. The procedure is similar. Solving a new equation, we obtain its general integer solution:

$$(p_2) \quad k_{i_2}^{(1)} = f_{i_2}^{(2)}(k_1^{(2)}, \dots, k_{n-2}^{(2)}), \quad i_2 = 1, n-1$$

where all the functions $f_{i_2}^{(2)}$ are linear, with integer number coefficients. (If, along this algorithm we come across an equation which does not admit integer solutions, then, the initial system does not admit integer solution. Stop.)

In the case that all the solved equations admitted integer systems at step (j) , $1 \leq j \leq r$, (r being of the same rank as the matrix associated to the system) then:

$$(j-1) \quad (j) \quad (j) \quad (j) \quad \dots$$

$$(p_j) \quad k_{i_j} = f_{i_j} (k_1, \dots, k_{n-j}), \quad i_j = 1, n-j+1,$$

$f_{i_j}^{(j)}$ are linear functions with integer number coefficients.

Finally, after r steps, and if all the solved equations admitted integer solutions, we get to the integer solution of one equation with $n-r+1$ unknown variables.

The system will accept integer solutions if and only in this last equation will have integer solutions. If it does, let the general integer solution of it be:

$$(p_r) \quad k_{i_r}^{(r-1)} = f_{i_r}^{(r)} (k_1, \dots, k_{n-1}), \quad i_r = 1, n-r+1,$$

where all $f_{i_r}^{(r)}$ are linear functions with integer number coefficients.

Now the reverse way follows.

We introduce the values of $k_{i_r}^{(r-1)}$, $i_r = 1, n-r+1$

at step (p_r) in the values of $k_{i_{(r-1)}}^{(r-2)}$, $i_{r-1} = 1, n-r+2$ from step (p_{r-1}) .

It follows:

$$= \frac{k_i^{(r-1)} f_i^{(r)}(f_1^{(r)}, \dots, k_{n-r}^{(r)}, \dots, f_{n-r+1}^{(r)}(k_1^{(r)}, \dots, k_{n-r}^{(r)}))}{r-1}$$

from which it follows that g_i are linear functions with $(r-1)$ integer number coefficients.

Then follow those from (p_{r-2}) in (p_{r-e}) , and so on, until we introduce the values obtained at step (p_2) in those from the step (p_1) . It will follow:

$$x_{i_1} = g_i^{(1)}(k_1, \dots, k_{n-r}) \quad \text{notation} \quad g_{i_1}(k_1, \dots, k_{n-r}),$$

with all g_{i_1} , most obviously, linear functions with integer number coefficients (the notation was made for simplicity and aesthetical aspects). This is, thus, the general integer solution, of the initial system.

The correctness of algorithm 1. The algorithm is finite because it has r steps on the first way and $r-1$

steps on the reverse. ($r < +4$). Obviously, if one equation of one system does not accept (integer number) solutions then the system does not have either.

Writing S for the initial system and S_j the system resulted from step (p_j) , $1 \leq j \leq r-2$, it follows that passing from (p_j) to (p_{j+1}) we pass from a system S_j to a system S_{j+1} equivalent from the viewpoint of the integer

number solution, i.e., $k_{i_j}^{(j-1)} = t_{i_j}^0$, $i_j = 1, n-j+1$ which is a particular integer solution of the system S_j if and only

if $k_{i_{j+1}}^{(j)} = h_{i_{j+1}}^0$, $i_{j+1} = 1, n-j$ is a particular integer

solution of the system S_{j+1} where $h_{i_{j+1}}^0 = f_i^{(j+1)}(t_1, \dots, t_{n-j+1})$,

$i_{j+1} = 1, n-j$. Hence, their general integer solutions are

also equivalent (considering these substitutions). So that, in the end, the solving of the initial system S is equivalent with the solving of the equation (of the system

consisting of one equation) S_{r-1} with integer number coefficients. It follows that the system S admits integer

number solution if and only if all the systems S_j admit integer number solution, $1 \leq j \leq r-1$.

Example 1. By means of algorhythmus 1, let us

calculate the integer number solution of the system:

$$(S) \quad \begin{aligned} 5x - 7y - 2z + 6w &= 6 \\ 9 - 4x + 6y - 3z + 11w &= 0 \end{aligned}$$

Solution: We solve the first integer number equation. We

obtain the general integer solution (see [4] or [5]):

$$(p_1) \quad \begin{aligned} x &= t_1 + 2t_2 \\ y &= t_1 \\ z &= -t_1 + 5t_2 + 3t_3 - 3 \\ w &= t_3, \end{aligned}$$

where $t_1, t_2, t_3 \in \mathbb{Z}$.

Substituting in the second, we get the system:

$$(S_1) \quad 5t_1 - 23t_2 + 2t_3 + 9 = 0$$

Solving this integer equation we obtain its general integer solution:

$$(p_2) \quad \begin{aligned} t_1 &= k_1 \\ t_2 &= k_1 + 2k_2 + 1 \\ t_3 &= 9k_1 + 23k_2 + 7, \end{aligned}$$

where $k_1, k_2 \in \mathbb{Z}$.

The reverse way. Substituting (p_2) in (p_1) we obtain:

$$\begin{aligned} x &= 3k_1 + 4k_2 + 2 \\ y &= k_1 \\ z &= 31k_1 + 79k_2 + 23 \\ w &= 9k_1 + 23k_2 + 7, \end{aligned}$$

where $k_1, k_2 \in \mathbb{Z}$ which is the general integer solution of

the initial system (S) . Stop.

Algorithm 2

Input

A linear system (1) without all $a_{ij} = 0$.

Output

We decide on the possibility of an integer solution of this system. If it is possible, we obtain its general integer solution.

Method

1. $t = 1, h = 1, p = 1$

2. (A) Divide each equation by the maximal divisor of the coefficients of the unknown variables. If you do not get an integer quotient for at least one equation, then the system does not admit integer solutions. Stop.

(B) If there is an inequality in the system, then the system does not admit integer solutions. Stop.

(C) If the repetition of more equations occurs, keep one and if an equation is an identity, remove it from the system.

3. If there is (i_0, j_0) so that $|a_{i_0 j_0}| = 1$, then obtain the value of the variable x_{j_0} from the equation i_0 ; relation (T_t) . Substitute this relation (where possible) in the other equations of the system and in the relations (T_{t-1}) , (H_h) and (P_p) for all i , h and p . Consider $t := t + 1$, remove equation (i_0) from the system. If there is not such a pair, go on to step 5.

4. Does the system (left) have at least one unknown and variable? If it does, consider the new date and go on to step 2. If it does not, write the general integer solution of the system substituting k_1, k_2, \dots for all the variables from the right term of each expression which gives the value of the unknowns of the initial system. Stop.

5. Calculate $a = \min_{i, j_1, j_2} \{ |r|, a_{ij_1} / r \pmod{a_{ij_2}}, 0 < |r| < |a_{ij_2}| \}$, and determine the indices i, j_1, j_2 as well as the r for which this minimum can be calculated. (If there are more variants, choose one, arbitrarily.)

6. Write: $x_j = t_h \frac{a_{ij_1} - r}{a_{ij_2}}$, relation (H_h) .

Substitute this relation (where possible) in all the equations of the system and in the relations $(T_t), (H_h)$ and (P_p) for all t, h and p .

7. (A) If $a \dots 1$, consider $x_j := t_h, h := h + 1$

and

2

go on to step 2.

(B) If $a = 1$, then obtain the value of x_j from

1

from the equation (i); relation (P_p) .

Substitute this relation (where possible) in the

other equations of the system and in the relations (T_t) , (H_h) and (P_{p-1}) for all t, h and p .

Remove the equation (i) from the system.

Consider $h := h + 1$, $p := p + 1$ and go back to step 4.

The correctness of algorithm 2. Let the system (1) be.

Lemma 1. We consider the algorithm at step 5. Also,

let $M = \{ |r|, a_{ij} / r \pmod{a_{ij}}, 0 < |r| < |a_{ij}|, i, j_1, j_2 = 1, 2, 3, \dots \}$. Then $M \dots$.

Proof

Obviously, M is finite and $M \neq \emptyset$. Then, M has a minimum if and only if $M \dots$. We suppose, conversely, that $M = \emptyset$. Then $a_{ij_2} / 0 \pmod{a_{ij_2}}$, $\forall i, j_1, j_2$. It follows conversely as well that $a_{ij_2} / 0 \pmod{a_{ij_1}}$, $\forall i, j_1, j_2$. That is $|a_{ij_2}| = |a_{ij_1}|$, $\forall i, j_1, j_2$. We consider a_{i_0} arbitrary but fixed. It is clear that $(a_{i_0 j_1}, \dots,$

..., $a_{i_0 j_1} = a_{i_0 j_2} \dots 0$, $\forall j$ (because the algorithm has passed

through the substeps 2(B) and 2(C)). But, as it has also

passed through step 3, it follows that $|a_{i_0 j_1}| \dots 1$, $\forall j$, but

as it previously passed through step 2(A), it would result

that $|a_{i_0 j_1}| = 1$, $\forall j$. This contradiction shows that the assumption is false.

Lemma 2. Let $a_{i_0 j_1} \not\equiv r \pmod{a_{i_0 j_2}}$. Substitute $x_{j_2} =$

$$= t_h - \frac{a_{i_0 j_1} - r}{a_{i_0 j_2}} x_{j_1}$$
 in the system (A) obtaining the system

(B). Then, $x_j = x_{j_1}^0, j = 1, n$ is the particular integer solution of (A) if and only if $x_j = x_{j_1}^0, j \dots j_2$ and $t_h =$

$$= x_{j_1}^0 + \frac{a_{i_0 j_1} - r}{a_{i_0 j_2}} x_{j_2}^0$$
 is the particular integer solution of (B).

Lemma 3. Let $a_1 \dots 1$ and a_2 be obtained at step 5.

Then $0 < a_2 < a_1$.

Proof

It is sufficient to show that $a_1 < |a_{ij}|$, $\forall i, j$ because

in order to get a_2 step 6 is obligatory, when the coefficients of the new system are calculated, a_1 being equal to a coefficient from the new system (equality of modules), the coefficient on $(i_0 j_1)$.

Let a_{ij} with the property $|a_{ij}| \leq a_1$. Hence, $a_1 \geq a_{ij}$. Let a_{ij} with $|a_{ij}| > |a_{ij}|$ there is such an element because $|a_{i_0 j_m}|$ is the minimum of the

coefficients in the module and not all $|a_{ij}|$, $j = 1, n$ are

equal (conversely, it would result that (a_{ij}, \dots, a_{in}) is such an element because $|a_{i_0 j_m}|$ is the minimum of the coefficients in the module and not all $|a_{ij}|$, $j = 1, n$ are

$- a_{ij}$, $\forall j \in 1, n$; the algorithm passed through substep 2 $(A)_0$

has simplified each equation by the maximal co-divisor of

its coefficients; hence, it would follow that $|a_{i_0 j}| = 1$,

$\forall j = 1, n$ which, again, cannot be real because

the algorithm has also passed through step 3). Of the

coefficients $a_{i,j}$ we choose one with the property $a_{i,j} \neq 0$

... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$...

...; $a_{i,j}$ there is such an element (contrary, it would result

$\begin{matrix} 0 & m \\ 0 & s \end{matrix}$... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$) - $|a_{i,j}|$, but the algorithm has also passed

$\begin{matrix} 0 & m \\ 0 & s \end{matrix}$... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$

through step 2(A) and it would mean that $|a_{i,j}| = 1$

$\begin{matrix} 0 & m \\ 0 & s \end{matrix}$... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$

which contradicts step 3 through which the algorithm has also passed).

Considering $q_0 = \frac{a_{i,j}}{-a_{0,s_0}} \in \mathbb{Z}$ and $r = a_{i,j} - q_0 a_{0,j_m} \in \mathbb{Z}$, we have $a_{i,j} \equiv r_0 \pmod{a_{0,j_m}}$ and $0 < |r_0| < |a_{i,j}| < |a_{i,j}| \leq a_1$. We have, thus, obtained an

$\begin{matrix} 0 & m \\ 0 & s \end{matrix}$... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$... $\begin{matrix} 0 & m \\ 0 & s \end{matrix}$

r_0 with $|r_0| < a_1$, which is in contradiction with the very

definition of a_1 . Thus, $a_1 < |a_{ij}|$, $\forall i, j$.

Lemma 4. Algorithm 2 is finite.

Proof

The functioning of the algorithm is meant to transform a linear system of m equations and n unknowns into one of $m_1 \times n_1$, with $m_1 < m$, $n_1 \leq n$ and, thus, successively into a final linear equation with $n - r + 1$ unknowns (where r is the rank of the associated matrix).

This equation is solved by means of the same algorithm (which works as [5]). The general integer solution of the system will depend on the $n - 1$ integer number independent parameters (see [6]--similar properties can be established also for the general integer solution of the linear system). The reduction of equations occurs at steps 2, 3 and substep 7(B). Steps 2 and 3 are obvious and, hence, trivial; they can reduce the equations of the system (or even put an end to it) but only under particular conditions. The most important case finds its solution at step 7(B), which always reduces one equation of the system. As the number of equations is finite, we come to solve a single integer number equation. We also have to show that the transfer from one system $m_i \times n_i$ to another $m_{i+1} \times n_{i+1}$ is made in a finite interval of time: by steps 5 and 6 permanent substitution of variables are made until we get to $a = 1$ (we get to $a = 1$ because, according to lemma 3, all $a - s$ are positive integer numbers and form a strictly decreasing row).

Theorem of correctness. Algorithm 2 correctly calculates the general integer solution of the linear system.

Proof

Algorithm 2 is finite according to lemma 4. Steps 2 and 3 are obvious (see also [4], [5]). Their part is to simplify calculus as much as possible. Step 4 tests the finality of the algorithm; the substitution with the parameters k_1, k_2, \dots has systemization and aesthetic reasons. The variables t, h, p are counter variables (started at step 1) and they are meant to count the relations of the type T, H, P (numbering required by the substitutions at steps 3, 6 and substep 7(B)); h also counts the new (auxiliary) variables introduced in the moment of decomposition of the system. The substitution from step 6 does not affect the general integer solution of the system (it follows from lemma 2). Lemma 1 shows that at step 5 there is always a , because $\dots M \neq N^*$.

The algorithm performs the transformation of a system $m_i x_n i$ into another, $a_{i+1} x_{n+1}$, equivalent to it, preserving the general solution (taking into account, however, the substitutions) (see also lemma 2).

Example 2. Calculate the integer solution of:

$$\begin{aligned}
 -12x - 7y + 9z &= 12 \\
 -5y + 8z + 10w &= 0 \\
 0z + 0w &= 0 \\
 15x + 21z + 69w &= 3
 \end{aligned}$$

Solution

We apply algorithm 2 (we purposely looked for an example to be passed through all the steps of this algorithm:

1. $t = 1, h = 1, p = 1$
2. (A) The fourth equation becomes: $5x + 7z + 23w = 1$
- (B) --
- (C) Equation 3 is removed.
3. No; go on to step 5.
5. $a = 2$ and $i = 1, j_1 = 2, j_2 = 3$ and $r = 2$.
6. $z = t_1 + y$, the relation (H_1) . Substituting it in the system:

$$\begin{aligned} -12x + 2y + 9t_1 &= 12 \\ 3y + 8t_1 + 10w &= 0 \\ 5x + 7y + 7t_1 + 23w &= 1 \end{aligned}$$

7. $a \dots 1$; consider $z: = t_1, h: = 2$ and go back to step 2.

2. --
3. No. Step 5.
5. $a = 1$ and $i = 2, j_1 = 4, j_2 = 2$ and $r = 1$.

6. $y = t_2 - 3w$, the relation (H_2) . Substituting in the system:

$$-12x + 2t_2 + 9t_1 - 6w = 12$$

$$3t_2 + 8t_1 + w = 0$$

$$5x + 7t_2 + 7t_1 + 2w = 1$$

Substituting it in relation to (H_1) , we get:

$$z = t_1 + t_2 - 3w, \text{ relation } (H_1)'.$$

7. $w = -3t_2 - 8t_1$, relation (P_1) .

Substituting it in the system, we get:

$$-12x + 20t_2 + 57t_1 = 12$$

$$5x + t_2 - 9t_1 = 1$$

Substituting it in the other relations, we get:

$$z = 10t_2 + 25t_1, \text{ (H}_1\text{) }'';$$

$$y = 10t_2 + 24t_1, \text{ (H}_2\text{) }'';$$

$h: = 3$, $p: = 2$ and go back to step 4.

4. Yes

2. --

3. $t_2 = 1 - 5x + 9t_1$, relation (T_1) .

Substituting it (where possible) we get:

$$\{-112x + 237t_1 = -8 \text{ (the new system)};$$

$$z = 10 - 50x + 115t_1, \text{ (H}_1\text{) }'''$$

$$y = 10 - 50x + 114t_1, \quad (H_2)''$$

$$w = -3 + 15x - 35t_1, \quad (P_1)'$$

Consider $t_1 = 2$; go on to step 4.

4. Yes. Go back to step 2. (From now on algorithm 2 works similarly with that from [5], with the only difference that the substitutions must also be made in the relations obtained up to this point).
2. --
3. No. Go on to step 5.
5. $a = 13$ (one three) and $i = 1$, $j_1 = 2$, $j_2 = 1$ and $r = 13$.
6. $x = t_3 + 2t_1$, relation (H_3) .

After substitution we get:

$$-112t_3 + 13t_1 = -8 \quad (\text{the system})$$

$$z = 10 - 50t_3 + 15t_1, \quad (H_1)^{IV};$$

$$y = 10 - 50t_3 + 14t_1, \quad (H_2)'''$$

$$w = -3 + 15t_3 - 5t_1, \quad (P_1)''$$

$$t_2 = 1 - 5t_3 - t_1, \quad (T_1)'$$

7. $x_1 = t_3$, $h_1 = 4$ and go on to step 2.
2. --
3. No, go on to step 5.

5. $a = 5$ and $i = 1$, $j_1 = 1$, $j_2 = 2$ and $r = 5$

6. $t_1 = t_4 + 9t_3$, relation (H_4) .

Substituting it, we get: $5t_3 + 13t_4 = -8$ (the system).

$$z = 10 + 85t_3 + 15t_4, \quad (H_1)^V ;$$

$$y = 10 + 76t_3 + 14t_4, \quad (H_2)^{IV} ;$$

$$x = 19t_3 + 2t_4, \quad (H_3) ' ;$$

$$w = -3 - 30t_3 - 5t_4, \quad (P_1) ' ' ' ;$$

$$t_2 = 1 - 14t_3 - t_4, \quad (T_1) ' ' ;$$

7. $t_1 := t_4$, $h := 5$ and go back to step 2.

2. --

3. No; step 5.

5. $a = 2$ and $i = 1$, $j_1 = 2$, $j_2 = 1$ and $r / -2$.

6. $t_3 = t_5 - 3t_4$, relation (H_5) . After substitution,

we get:

$$5t_5 - 2t_4 = -8 \quad (\text{the system})$$

$$z = 10 + 85t_5 - 240t_4, \quad (H_1)^{VI} ;$$

$$y = 10 + 76t_5 - 214t_4, \quad (H_2)^V ;$$

$$x = 19t_5 - 55t_4, \quad (H_3)^{IV} ;$$

$$w = -3 - 30t_5 + 85t_4, \quad (P_1)^{IV} ;$$

$$t_2 = -1 - 14t_5 + 41t_4, \quad (T_1) ' ' ' ;$$

$$t_1 = 9t_5 + 26t_4, \quad (H_4) ' ;$$

7. $t_3 := t_6$, $h := 6$ and go back to step 2.

2. --

3. No; step 5.

5. $a = 1$ and $i = 1$, $j_1 = 1$, $j_2, r = 1$.

6. $t_4 = t_6 + 2t_5$, relation (H_6) . After substitution,

we get:

$$t_5 - 2t_6 = -8 \quad (\text{the system})$$

$$z = 10 - 395t_5 - 240t_6, \quad (H_1)^{VII};$$

$$y = 10 - 392t_5 - 214t_6, \quad (H_2)^{VI};$$

$$x = -91t_5 - 55t_6, \quad (H_3)''';$$

$$w = -3 + 140t_5 + 85t_6, \quad (P_1)^V;$$

$$t_2 = 1 + 68t_5 + 41t_6, \quad (T_1)^{IV};$$

$$t_1 = -43t_5 - 26t_6, \quad (H_4)'';$$

$$t_3 = -5t_5 - 3t_6, \quad (H_5)';$$

7. $t_5 = 2t_6 - 8$, relation (P_2) . Substituting it in

the system, we get: $0 = 0$.

Substituting it in the other relations, it

follows:

$$z = -1030t_6 + 3170$$

$$y = -918t_6 + 2826$$

```

x      = - 237t6 + 728
w      = 365t6 - 1123
t2    = 177t6 - 543
t1    = 112t6 + 344
                                relations of no
importance
t3    = 13t6 + 40
t4    = 5t6 - 16

```

Consider h: = 7, p: = 3 and go back to step 4.

t₆ 0 Z

4. No. The general integer solution of the system

is:

$$\begin{aligned}
 x &= -237k_1 + 728 \\
 y &= -918k_1 + 2826 \\
 z &= 1030k_1 + 3170 \\
 w &= 365k_1 - 1123
 \end{aligned}$$

where k₁ is an integer number parameter.

Stop.

Algorithm 3

Input

A linear system (1).

Output

We decide on the possibility of an integer solution of this system. If it is possible, we obtain its general integer solution.

Method

1. Solve the system in R^n . If it does not have solutions in R^n , it does not have solutions in Z^n either. Stop.
2. $f = 1, t = 1, h = 1, g = 1$
3. Write the value of each main variable x_i under the form:

$$(E_{f,i})_i: x_i = \sum_j q_{ij} x_j + q_i + \sum_j r_{ij} x_j + r_i / a_i,$$

with all $q_{ij}, q_i, r_{ij}, r_i, a_i$ in Z so that all of the right term are integer number variables: either of the secondary variables of the system

or other new variables introduced with the algorithmus). For all i , we write r_{ij_f} / a_i .

4. $(F_{f,i})_i: \sum_j r_{ij} x_j - r_{i,j_f} y_{f,i} + r_i = 0$ where $(Y_{f,i})_i$

are auxiliary integer number variables. We remove all the equations $(F_{f,i})$ which are identities.

5. Does at least one equation $(F_{f,i})$ exist? If it does not, write the general integer solution of the system substituting k_1, k_2, \dots for all the

variables from the right term of each expression

representing the value of the initial unknowns of the system. Stop.

6. (A) Divide each equation $(F_{f,i})$ by the maximal co-divisor of the coefficients of their unknowns. If the quotient is not an integer number for at least one i_0 then the system does not admit integer solutions.

Stop.

(B) Simplify--as in m--all the fractions from the relations $(E_{f,i})_i$.

7. Does it exist $r_{i_0 j_0}$ with the module 1? If it does not, go on to step 8.
 If it does, find the value of x_j from the equation $(F_{f,i})$; write (T_t) for this relation and

0

substitute it (where it is possible) in the relations $(E_{f,i})$, (T_{t-1}) , (H_h) , (G_g) for all i , t ,

h and g. Remove the equation $(F_{f,i})$.
Consider

0

$f := f + 1$, $t := t + 1$ and go back to step 3.

8. Calculate $a = \min_{i, j_1, j_2} \{ |r|, r_{ij_1} / r \pmod{r_{ij_2}}, 0 < |r| < |r_{ij_2}| \}$ and determine the indices i_m , j_1 , j_2 as well as the r for which this minimum can be obtained. (When there are more variants,

choose only one).

9. (A) Write $x_j = z_h - \frac{a_{i_m j_1} - r}{a_{j_m j_2}}$, where z_h is

new integer variable; relation (H_h) .

(B) Substitute the letter (where possible) in the relations $(E_{f,i})$, $(F_{f,i})$, (T_t) , (H_{h-1}) , (G_g) for all i , t , h and g .

(C) Consider $h := h + 1$.

10. (A) If $a \dots 1$, go back to step 4.

(B) If $a = 1$, calculate the value of the variable x_j from the equation $(F_{f,i})$;

relation (G_g) . Substitute it (where possible) in the relations $(E_{f,i})$, (T_t) , (H_h) , (G_{g-1}) for all i, t, h and g . Remove the equation $(F_{f,i})$. Consider $g := g + 1$, $f := f + 1$ and go back to step 3.

The correctness of algorithm 3

Lemma 5. Let i be fixed. Then $(\sum_{j=n_1}^{n_2} r_{ij}x_j + r_i)/a_i$

(with all r_{ij} , r_i , a_i , n_1 , n_2 being integers, $n_1 \leq n_2$, $a_i \geq 0$)

and all x_j being integer variables) can have integer values

if and only if $(r_{i n_1}, \dots, r_{i n_2}, a_i)/r_i$.

Proof

The fraction from the enunciation can have integer

values if and only if there is a $z \in \mathbb{Z}$ so that $(\sum_{j=n_1}^{n_2} r_{ij}x_j + r_i)/a_i = z$] $\sum_{j=n_1}^{n_2} r_{ij}x_j - a_i z + r_i = 0$ which is a linear

equation. This equation admits integer solution] $(r_{i n_1}, \dots, r_{i n_2}, a_i) \mid r_i$.

Lemma 6. The algorithm is finite. It is true. The algorithm can stop at steps 1, 5 or substep 6(A). (It rarely happens to stop at step 1). An equation after another are gradually eliminated at step 4 and especially 7 and 10(B) ($F_{f,i}$)--the number of equations is finite. If at steps 4 and 7 the elimination of equation may occur only in special cases, elimination of equations at the substep 10(B) is always true because, through steps 8 and 9 we get to $a = 1$ (see [5]) or even lemma 4 (from the correctness of algorithm 2). Hence, the algorithm is finite.

Theorem of Correctness. The algorithm 3 correctly calculates the general integer solution of the system (1).

Proof

The algorithm is finite according to lemma 6. It is obvious that if the system does not have solutions in R^n it does not have in Z^n either, because $Z^n \subset R^n$ (step 1). The variables f, t, h, g are counter variables and are meant to number the relations of the type E, F, t, H and G, respectively. They are used to distinguish between the relations and make the necessary substitutions (step 2). Step 3 is obvious. All the coefficients of the unknowns being integers, each main variable x_i will be written:

$$\mathbf{x}_i = \left(\sum_j c_{ij} \mathbf{x}_j + c_i \right) / a_i ,$$

which can assume the form and conditions required in this step. Step 4 is obtained from 3 by writing each fraction equal to an integer variable $y_{f,i}$ (this being $x_i \in \mathbb{Z}$). Step 5 is very close to the end. If there is no fraction among all $(E_{f,i})$ it means that all the main variables x_i already have values in \mathbb{Z} , while the secondary variables of the system can be arbitrary in \mathbb{Z} , or can be obtained from the relations T, H or G (but these have only integer expressions because of their definition and because only integer substitutions are made). The second assertion of this step is meant to systematize the parameters and renumber; it could be left out but aesthetic reasons dictate its presence. According to lemma 5 the step 6(A) is correct. (If a fraction depending on certain parameters (integer variables) cannot have values in \mathbb{Z} , then the main variable which has in the value of its expression such a fraction cannot have values in \mathbb{Z} either; hence, the system does not admit integer systems). This 6(A) also has a simplifying role. The correctness of step 7, trivial as it is, also results from [4], and the steps

8-10 from [5] or even from algorithmus 2 (lemma 4).

The initial system is equivalent to the "system" from step 3 (in fact, $(E_{f,i})$ as well, can be considered a system). So, the general integer solution is preserved (the changes of variables do not prejudice it (see [4], [5], and also lemma 2 from the correctness of algorithm 2)). From a system $m_i x_{n_i}$ we form an equivalent system $m_{i+1} x_{n_{i+1}}$ with $m_{i+1} < m_i$ and $n_{i+1} < n_i$. This algorithm works similarly to algorithm 2.

Example 3. Employing algorithm 3, find an integer solution of the following system:

$$\begin{aligned} 3x_1 + 4x_2 + 22x_4 - 8x_5 &= 25 \\ 6x_1 + \quad \quad \quad + 46x_4 - 12x_5 &= 2 \\ 4x_2 + 3x_3 - \quad x_4 + 9x_5 &= 26 \end{aligned}$$

Solution

1. Common solving in \mathbb{R}^3 , it follows:

$$x_1 = \frac{23x_4 - 6x_5 - 1}{-3}$$

$$x_2 = \frac{x_4 + 2x_5 + 24}{4}$$

$$x_3 = \frac{11x_5 + 2}{3}$$

2. $f = 1, t = 1, h = 1, g = 1$

3.

$$x_1 = -7x_4 + 2x_5 + \frac{2x_4 - 1}{-3} \quad (E_{1,1})$$

$$x_2 = 6 + \frac{x_4 + 3x_5}{4} \quad (E_{1,2})$$

$$x_3 = -4x_5 + \frac{x_5 + 2}{3} \quad (E_{1,3})$$

$$4. \quad 2x_4 + 3y_{11} - 1 = 0 \quad (F_{1,1})$$

$$x_4 + 2x_5 - 4y_{12} = 0 \quad (F_{1,2})$$

$$x_5 - 3y_{13} + 2 = 0 \quad (F_{1,3})$$

5. Yes.

6. --

7. Yes: $|r_{35}| = 1$. Then $x_5 = 3y_{13} - 2$, the relation (T_1) . Substituting it in the others, we get:

$$x_1 = -7x_4 + 6y_{13} - 4 + \frac{2x_4 - 1}{-3} \quad (E_{1,1})$$

$$x_2 = 6 + \frac{x_4 + 6y_{13} - 4}{4} \quad (E_{1,2})$$

$$x_3 = -12y_{13} + 8 + \frac{3y_{13} - 2 + 2}{3} \quad (E_{1,3})$$

Remove the equation $(F_{1,3})$.

Consider $f: = 2$, $t: = 2$; go back to step 3.

$$3. \quad x_1 = -7x_4 + 6y_{13} - 4 + \frac{2x_4 - 1}{-3} \quad (E_{2,1})$$

$$x_2 = y_{13} + 5 + \frac{x_4 + 2y_{13}}{4} \quad (E_{2,2})$$

$$x_3 = -11y_{13} + 8 \quad (E_{2,3})$$

$$4. \quad 2x_4 + 3y_{21} - 1 = 0 \quad (F_{2,1})$$

$$x_4 + 2y_{13} - 4y_{22} = 0 \quad (F_{2,2})$$

5. Yes.

6. --

7. Yes. $|r_{24}| = 1$. We obtain $x_4 = -2y_{13} + 4y_{22}$, relation (T_2) . Substituting it in the others we get:

$$x_1 = -28y_{22} + 20y_{13} - 4 + \frac{-4y_{13} + 8y_{22} - 1}{-3} \quad (E_{2,1})'$$

$$x_2 = y_{22} + y_{13} + 5 \quad (E_{2,2})'$$

$$x_3 = -11y_{13} + 8 \quad (E_{2,3})'$$

Remove the equation (F_{22}) .

Consider $f: = 3$, $t: = 3$ and go back to step 3.

$$3. \quad x_1 = -22y_{13} - 30y_{22} - 4 + \frac{-2y_{13} + 2y_{22} - 1}{-3} \quad (E_{3,1})$$

$$x_2 = y_{13} + y_{22} + 5 \quad (E_{3,2})$$

$$x_3 = -11y_{13} + 8 \quad (E_{3,3})$$

$$4. \quad 2y_{13} + 2y_{22} + 3y_{31} - 1 = 0 \quad (F_{3,1})$$

5. Yes.

6. --

7. No.

8. $a = 1$, and $i_m = 1$, $j_1 = 31$, $j_2 = 22$ and $r = 1$.

9. (A) $y_{22} = z_1 - y_{31}$, relation (H_1) .

(B) Substituting it in the others we get:

$$x_1 = -22y_{13} - 30z_1 + 30y_{31} - 4 + \frac{-2y_{13} + 2z_1 - 2y_{31} - 1}{-3} \quad (E_{3,1})'$$

$$x_2 = y_{13} + z_1 - y_{31} + 5 \quad (E_{3,2})'$$

$$x_3 = -11y_{13} + 8 \quad (E_{3,3})'$$

$$2y_{13} + 2z_1 + y_{31} - 1 = 0 \quad (F_{3,1})'$$

$$x_4 = -2y_{13} + 4z_1 - 4y_{31} \quad (T_2)'$$

(C) Consider $h: = 2$

10. (B) $y_{31} = 1 - 2y_{13} - 2z_1$, relation (G_1) .

Substituting it in the others we get:

$$x_1 = -40y_{13} - 92z_1 + 27 \quad (E_{3,1})''$$

$$x_2 = 3y_{13} + 3z_1 + 4 \quad (E_{3,2})''$$

$$x_3 = -11y_{13} + 8 \quad (E_{3,3})''$$

$$x_4 = 6y_{13} + 12z_1 - 4 \quad (T_2)''$$

$$y_{22} = 2y_{13} + 3z_1 - 1 \quad (H_1)'$$

Remove the equation $(F_{3,1})$.

Consider $g: = 2$, $f: = 4$ and go back to step 3.

$$3. \quad x_1 = -40y_{13} - 92z_1 + 27 \quad (E_{4,1})$$

$$x_2 = 3y_{13} + 3z_1 + 4 \quad (E_{4,2})$$

$$x_3 = -11y_{13} + 8 \quad (E_{4,3})$$

4. --

5. No. The general integer solution of the initial system is:

$$x_1 = -40k_1 - 92k_2 + 27, \quad \text{from } (E_{4,1})$$

$$x_2 = 3k_1 + 3k_2 + 4, \quad \text{from } (E_{4,2})$$

$$x_3 = -11k_1 + 8, \quad \text{from } (E_{4,3})$$

$$x_4 = 6k_1 + 12k_2 - 4, \quad \text{from } (T_2)''$$

$$x_5 = 3k_1 - 2, \quad \text{from } (T_1)$$

where $k_1, k_2 \in \mathbb{Z}$.

Algorithm 4

Input

A linear system (1) with not all $a_{ij} = 0$.

Output

We decide on the possibility of an integer solution of this system. If it is possible, we obtain its general integer solution.

Method

1. $h = 1, v = 1.$
2. (A) Divide every equation i by the maximal divisor of the coefficients of the unknowns. If the quotient is not an integer for at least one i_0 , then the system does not admit integer solutions. Stop.
 - (B) If there is an inequality in the system, then it does not admit integer solutions.
 - (C) In case of repetition, retain only one equation of that kind.
 - (D) Remove all the equations which are identities.
3. Calculate $a = \min_{i,j} \{ |a_{ij}|, a_{ij} \dots 0 \}$ and determine the indices i_0, j_0 for which this minimum can be obtained. (If there are more variants, choose one, at random.)
4. If $a \dots 1$, go on to step 6.
If $a = 1$, then:
 - (A) Calculate the value of the variable x_{j_0} from the equation i_0 ; write this relation (V_v) .
 - (B) Substitute this relation (where possible) in all the equations of the system as well

as in the relations (V_{v-1}) , (H_h) for all v and h .

(C) Remove the equation i_0 from the system.

(D) Consider $v: = v+1$.

5. Does at least one equation exist in the system?

(A) If it does not, write the general integer solution of the system substituting k_1, k_2, \dots for all the variables from the right term of each expression representing the value of the initial unknowns of the system.

(B) If it does, considering the new data, go back to step 2.

6. Write all $a_{i_0 j}, j \dots j_0$ and b_{i_0} under the form:

$$a_{i_0 j} = a_{i_0 j_0} q_{i_0 j} + r_{i_0 j}, \text{ with } |r_{i_0 j}| < |a_{i_0 j}| ;$$

$$b_{i_0} = a_{i_0 j_0} q_{i_0} + r_{i_0}, \text{ with } |r_{i_0}| < |a_{i_0 j_0}| .$$

7. Write $x_{j_0} = - \sum_{j \dots j_0} q_{i_0 j} x_j + q_{i_0} + t_h$, relation (H_h) . Substitute (where possible) this relation in all the equations of the system as well as in the relations (V_v) , (H_h) for all v and h .

8. Consider

$x_{j_0} := t_h, h := h + 1,$

$a_{i_0 j} := r_{i_0 j}, j \dots j_0,$

$a_{i_0 j_0} := \pm a_{i_0 j_0}, b_{i_0} := +r_{i_0},$

and go back to step 2.

The Correctness of Algorithm 4

This algorithm extends the algorithm from [4] (integer solutions of equations to integer solutions of linear systems). The algorithm was strictly demonstrated in our previous article; the present one introduces a new cycle--having as cycling variable the number of equations of the system--the rest remaining unchanged; hence, the correctness of algorithm 4 is obvious.

Discussion

1. The counter variables h and v count the relations H and V , respectively, differentiating them (to enable the substitutions);
2. Step 2 $(A + B) + (C)$ is trivial and is meant to simplify the calculus (as algorithm 2);
3. Substep 5(A) has aesthetic function (as all the

algorithms described). Everything else has been proven in the previous chapters (see [4], [5], and algorithm 2).

Example 4. Let us use algorithm 4 to calculate the integer solution of the following linear system:

$$\begin{array}{rcl} 3x_1 & - & 7x_3 + 6x_4 = -2 \\ 4x_1 + 3x_2 & + & 6x_4 - 5x_5 = 19 \end{array}$$

Solution

1. $h = 1, v = 1$
2. --
3. $a = 3$ and $i = 1, j = 1$
4. $3 \dots 1.$ Go on to step 6.
6. So,

$$\begin{array}{rcl} -7 & = & 3 \cdot (-3) + 2 \\ 6 & = & 3 \cdot 2 + 0 \\ -2 & = & 3 \cdot 0 - 2 \end{array}$$

7. $x_1 = 3x_3 - 2x_4 + t_1$, relation (H_1) . Substituting it in the second equation we get:

$$4t_1 + 3x_2 + 12x_3 - x_4 - 5x_5 = 19$$

8. $x_1: = t_1, h: = 2, a_{12}: = 0, a_{13}: = +2, a_{14}: = 0,$

$$a_{11} = +3, b = -2$$

Go back to step 2.

2. The equivalent system was written:

$$+ 3t_1 + 3x_3 = -2$$

$$4t_1 + 3x_2 + 12x_3 - x_4 - 5x_5 = 19$$

3. $a = 1, i = 2, j = 4$

4. $l = 1$

(A) Then: $x_4 = 4t_1 + 3x_2 + 12x_3 - 5x_5 - 19$,
relation (V_1) .

(B) Substituting it in (H_1) , we get:

$$x_1 = -7t_1 - 6x_2 - 21x_3 + 10x_5 + 38, \quad (H_1)$$

(C) Remove the second equation of the system.

(D) Consider: $v = 2$.

5. Yes. Go back to step 2.

2. The equation $+ 3t_1 + 2x_3 = -2$ is left.

3. $a = 2$ and $i = 1, j = 3$

4. $2 \dots 2$, go to step 6.

6. $+ 3 = + 2 ! 2 - 1$

$$- 2 = + 2 (-1) + 0$$

7. $x_3 = -2t_1 + t_2 - 1$, relation (H_2) .

Substituting it in $(H_1)', (V_1)$, we get:

$$x_1 = 35t_1 - 6x_2 - 21t_2 + 10x_5 + 59 \quad (H_1)'''$$

$$x_4 = -20t_1 + 3x_2 + 12t_2 - 5x_5 - 31 \quad (V_1)'.$$

8. $x_3: = t_2, h: = 3, a_{11}: = -1, a_{13}: = +2, b_1: = 0$
 (the others being all = 0). Go back to step 2.

2. The equation $-5_1 + 2t_2 = 0$ was obtained.

3. $a = 1$, and $i = 1, j = 1$

4. $1 = 1$

(A) Then, $t_1 = 2t_2$, relation (V_2) .

(B) After substitution, we get:

$$x_1 = 49t_2 - 6x_2 + 10x_5 + 59 \quad (H_1)'''$$

$$x_4 = -28t_2 + 3x_2 - 5x_5 - 31 \quad (V_1)'';$$

$$x_3 = -3t_2 - 1 \quad (H_2)';$$

(C) Remove the first equation from the system.

(D) $v: = 3$

5. No. The general integer solution of the initial system is:

$$x_1 = 49k_1 - 6k_2 + 10k_3 + 59$$

$$x_2 = k_2$$

$$x_3 = -3k_1 - 1$$

$$x_4 = -28k_1 + 3k_2 - 5k_3 - 31$$

$$x_5 = k_3$$

where $(k_1, k_2, k_3) \in \mathbb{Z}^3$.

Stop.

Algorithm 5

Input

A linear system (1).

Output

We decide on the possibility of a integer solution of this system. If it is possible, we obtain its general integer solution.

Method

1. We solve the common system in \mathbb{R}^n . If it does not have solutions in \mathbb{R}^n , then it does not have solutions in \mathbb{Z}^n either. Stop.
2. $f = 1, v = 1, h = 1$
3. Write the value of each main variable x_i under the form:

$$(E_{f,i})_i: x_i = \sum_j q_{ij}x_j + q_i + \sum_j r_{ij}x_j + r_i / a_i,$$

with all $q_{ij}, q_i, r_{ij}, r_i, a_i$ from \mathbb{Z} , so that all $|r_{ij}| < |a_i|$, $|r_i| < |a_i|$, $a_i \dots 0$ (where all x_j 's

of the right term are integer variables: either from the secondary variables of the system or the new variables introduced with the algorithm). For all i , we write $r_{i,j_f} / a_i$.

4. $(F_{f,i})_i: \sum_j r_{ij}x_j - r_{i,j} y_{f,i} + r_i = 0$, where $y_{f,i}$

are auxiliary integer variables. Remove all the equations $(F_{f,i})$ which are identities.

5. Does it exist at least one equation $(F_{f,i})$? If it does not, write the general integer solution of the system substituting k_1, k_2, \dots for all the variables of the right member of each expression representing the value of the initial unknowns of the system. Stop.

6. (A) Divide each equation $(F_{f,i})$ by the maximal co-divisor of the coefficients of their unknowns. If the quotient is not an integer for at least one i_0 , then the system does not admit integer solutions. Stop.

(B) Simplify--as previously ((A)) all the fractions in the relations $(E_{f,i})_i$.

7. Calculate $a = \min_{i,j} \{|r_{ij}|, r_{ij} \dots 0\}$, and determine the indices i_0, j_0 for which this minimum is

obtained.

8. If $a \dots 1$, go on to step 9.

If $a = 1$, then:

(A) Calculate the value of the variable x_{j_0} from the equation $(F_{f,i})$; write (V_v) for this relation.

(B) Substitute this relation (where possible) in the relations $(E_{f,i})$, (V_{v+1}) , (H_h) for all i , v and h .

(C) Remove the equation $(F_{f,i})$.

(D) Consider $v := v + 1$, $f := f + 1$ and go back to step 3.

9. Write all $r_{i_0 j}$, $j \dots j_0$ and r_{i_0} under the form:

$$r_{i_0 j} = \frac{a_{i_0}}{q_{i_0}} ! q_{i_0 j} + r_{i_0}, \text{ with } |r_{i_0 j}| < |\frac{a_{i_0}}{q_{i_0}}|;$$

$$r_{i_0} = \frac{a_{i_0}}{q_{i_0}} ! q_{i_0} + r_{i_0} \text{ with } |r_{i_0}| < |\frac{a_{i_0}}{q_{i_0}}|.$$

10. (A) Write $x_{j_0} = - \sum_{j \dots j_0} q_{i_0 j} ! x_j + q_{i_0} + t_h$, relation (H_h) .

(B) Substitute this relation (where possible) in all the relations $(E_{f,i})$, $(F_{f,i})$, (V_v) , (H_{h-1}) .

(C) Consider $h := h + 1$ and go back to step 4.

The correctness of the algorithm is obvious. It consists of the first part of algorithm 3 and the end part

of algorithm 4. Then, steps 1-6 and their correctness were discussed in the case of algorithm 3. The situation is similar with steps 7-10. (After calculating the real solution in order to calculate the integer solution, we resorted to the procedure from 5 and algorithm 5 was obtained.) It means that all these insertions were proven previously.

Example 5

Using algorithm 5, let us obtain the general integer solution of the system:

$$3x_1 + 6x_3 + 2x_4 = 0$$

$$4x_2 - 2x_3 - 7x_5 = -1$$

Solution

1. Solving in R^5 , we get:

$$x_1 = \frac{-6x_3 - 2x_4}{3}$$

$$x_2 = \frac{2x_3 + 7x_5 - 1}{4}$$

2. $f = 1, v = 1, h = 1$

$$\begin{aligned}
 3. \quad (E_{1,1}) \quad : \quad x_1 &= 2x_3 & + \frac{-2x_4}{3} \\
 (E_{1,2}) \quad : \quad x_2 &= x_5 & + \frac{2x_3+3x_5-1}{4}
 \end{aligned}$$

$$\begin{aligned}
 4. \quad (F_{1,1}) \quad : \quad -2x_4 - 3y_{11} &= 0 \\
 (F_{1,2}) \quad : \quad 2x_3 + 3x_5 - 4y_{12} - 1 &= 0
 \end{aligned}$$

5. Yes

6. --

7. $i = 2$ and $i_0 = 2$, $j_0 = 3$

8. 2 ... 1

9. $3 = 2 ! 1 + 1$

$-4 = 2 ! (-2)$

$-1 = 2 ! 0 - 1$

10. $x_3 = -x_5 + 2y_{12} + t_1$, relation (H_1) . After substitution:

$$\begin{aligned}
 (E_{1,1})' \quad : \quad x_1 &= 2x_5 - 4y_{12} - 2t_1 + \frac{-2x_4}{3} \\
 (E_{1,2})' \quad : \quad x_2 &= x_5 & + \frac{x_5+4y_{12}+2t_1-1}{4}
 \end{aligned}$$

$$(F_{1,2})' \quad : \quad x_5 + 2t_1 - 1 = 0$$

Consider $h = 2$ and go back to step 4.

$$4. \quad (F_{1,1})' \quad : \quad -2x_4 - 3y_{11} = 0$$

$$(F_{1,2})'': 2t_1 + x_5 - 1 = 0$$

5. Yes

6. --

7. $a = 1$ and $i_0 = 2, j_0 = 5$

(A) $x_5 = -2t_1 + 1$, relation (V_1)

(B) Substituting it, we get:

$$(E_{1,1})'': x_1 = -6t_1 + 2 - 4y_{12} + \frac{-2x_4}{3}$$

$$(E_{1,2})'': x_2 = -2t_1 + 1 + y_{12}$$

$$(H_1)' : x_3 = 3t_1 - 1 + 2y_{12}$$

(C) Remove the equation $(F_{1,2})$.

(D) Consider $v = 2, f = 2$ and go back to step
3.

$$3. (E_{2,1}) : x_1 = -6t_1 - 4y_{12} + 2 + \frac{-2x_4}{3}$$

$$(E_{2,2}) : x_2 = -2t_1 + y_{12} + 1$$

$$4. (F_{2,1}) : -2x_4 - 3y_{21} = 0$$

5. Yes

6. --

7. $a = 2$ and $i_0 = 1, j_0 = 4$

8. 2 ... 1

9. $-3 = -2 ! (1) - 1$

10. (A) $x_4 = -y_{21} + t_2$, relation (H_2) .

(B) After substitution, we get:

$$(E_{2,1})' : x_1 = -6t_1 - 4y_{12} + 2 + \frac{2y_{21}-2t_2}{3}$$

$$(F_{2,1})' : -y_{21} - 2t_2 = 0$$

Consider $h = 3$ and go back to step 4.

4. $(F_{2,1})' : -y_{21} - 2t_2 = 0$

5. Yes

6. --

7. $a = 1$ and $i_0 = 1$, $j_0 = 21$ (two, one).

(A) $y_{21} = -2t_2$, relation (v_2) .

(B) After substitution, we get:

$$(E_{2,1})' : x_1 = -6t_1 - 4y_{12} - 2t_2 + 2$$

$$(H_2)' : x_4 = 3t_2$$

(C) Remove the equation $(F_{2,1})$.

(D) Consider $v = 3$, $f = 3$ and go back to step 3.

3. $(E_{3,1}) : x_1 = -6t_1 - 4y_{12} - 2t_2 + 2$

$$(E_{3,2}) : x_2 = -2t_1 + y_{12} + 1$$

4. --

5. No. The general integer solution of the system is:

$$\begin{aligned}
 x_1 &= -6k_1 - 4k_2 - 2k_3 + 2, \quad \text{from } (E_{3,1}); \\
 x_2 &= -2k_1 + k_2 + 1, \quad \text{from } (E_{3,2}); \\
 x_3 &= 3k_1 + 2k_2 - 1, \quad \text{from } (H_1); \\
 x_4 &= \quad \quad \quad 3k_3, \quad \text{from } (H_2); \\
 x_5 &= -2k_1 + 1, \quad \text{from } (V_1);
 \end{aligned}$$

where $(k_1, k_2, k_3) \in \mathbb{Z}$.

Stop.

Note 1. Algorithms 3, 4 and 5 can be applied in the calculation of the integer solution of a linear equation.

Note 2. The algorithms, because of their form, are easy to introduce in the computer.

Note 3. It is up to the reader to decide on the algorithm to use. Good luck!

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